SUMMABILITY AND ASSOCIATIVE INFINITE MATRICES

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We consider the sequence-to-sequence matrix transformations $y = Ax$, where $A = (a_{nk})$, $x = \{x_k\}$, $y = \{y_n\}$,

$$y_n = A_n(x) = \sum_{k=0}^{\infty} a_{nk}x_k \qquad (n, k = 0, 1, 2, \cdots).$$

It is known that a matrix A is *conservative*, that is, Ax converges whenever x does, if and only if $\|A\| = \sup_n \Sigma_{k=0}^{\infty} |a_{nk}|$ is finite, $\lim_n \Sigma_{k=0}^{\infty} a_{nk}$ exists, and $\lim_n a_{nk}$ exists for $k = 0, 1, 2, \cdots$. If $A, B, C \cdots$ are conservative matrices with elements $a_{nk}, b_{nk}, c_{nk}, \cdots$, the column limits will be denoted by $a_k, b_k, c_k, \cdots$. A conservative matrix A is said to be *co-regular* if

$$\lim_n \sum_k a_{nk} - \sum_k a_k \neq 0.$$

Otherwise it is said to be *co-null.* Let e and e^n $(n = 0, 1, 2, \cdots)$ be the sequences defined respectively by $e_k = 1$ $(k = 0, 1, \cdots)$ and by $e_k^n = \delta_{nk}$ $(n, k = 0, 1, 2, \cdots)$. Let $\Delta = \{e^n: n = 0, 1, 2, \cdots\}$, and let Φ be the set consisting of the elements of Δ together with e. Let $H(\Delta)$ and $H(\Phi)$ be the linear hulls of Δ and Φ, respectively. The terms "basis" and "biorthogonal" will be used as in [1, pp. 106, 110].

We shall say that a matrix A is *associative* if $B(Ax) = BA(x)$ for all matrices B with $\|B\|$ finite and all x in the summability field C_A. Clearly a matrix A is associative if and only if

$$\sum_n t_n \sum_k a_{nk}x_k = \sum_k \sum_n t_n a_{nk} x_k \qquad \text{for all } x \in C_A \text{ and all } \{t_n\} \in (\gamma),$$

where (γ) denotes the set of sequences $\{t_n\}$ such that $\Sigma_{n=0}^{\infty} |t_n|$ converges. We shall show that if A is replaceable, that is, if there exists a regular matrix D such that $C_D = C_A$, then A is associative if and only if Φ is a basis for C_A.

Bases for the space C_A have been studied by Wilansky [3] and MacPhail [2]. A conservative matrix A is said to have *maximal inset* if $\Sigma\, a_k x_k$ converges for all x in C_A. A is said to have *propagation of maximal inset* (PMI) if $\Sigma\, b_k x_k$ converges for all x in C_A whenever B is a matrix such that $C_B = C_A$. Wilansky has shown that if A is a triangular co-regular matrix, then Φ is a basis for C_A if and only if A has PMI. MacPhail has shown that this statement is true if "triangular" is replaced by "reversible." We shall show that if A is an arbitrary co-regular matrix, then Φ is a basis for C_A if and only if A has PMI. Also, we shall give necessary and sufficient conditions that Δ be a basis for C_A.

LEMMA 1. *Let A be a co-regular matrix.* $\overline{H(\Phi)} = C_A$, *that is,* $H(\Phi)$ *is dense in* C_A, *if and only if, for each sequence* $\{b_n\}$ *such that* $\Sigma|b_n|$ *is convergent and*

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