DENSE SUBSETS IN THE SPACES l_p

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It is well known that the space l_p is defined, for $1 \leq p < \infty$, as the linear space of all sequences $x = \{\xi_k\}$ of scalars for which $\Sigma_{k=1}^{\infty} |\xi_k|^p$ is finite. If we set $\|x\| = (\Sigma_{k=1}^{\infty} |\xi_k|^p)^{1/p}$, we get a Banach space. Every linear continuous functional x^* in l_p is determined in one and only one way by a sequence $x^* = \{\alpha_k\}$, with

$$\sum_{k=1}^{\infty} |\alpha_k|^q < \infty \qquad \left(\frac{1}{p} + \frac{1}{q} = 1\right),$$

by means of the relation $x^*(x) = \Sigma_{k=1}^{\infty} \alpha_k \xi_k$.

If S *is a subset of* l_p, *and if the only linear continuous functional which vanishes on* S *is the null functional, then* S *determines a dense subspace of* l_p (see [1, p. 57], [5, p. 9], and [6, p. 61]).

Inspired by M. V. Subba Rao's paper [6], we obtain, by means of Dirichlet series, a number of propositions concerning dense linear subsets in l_p.

PROPOSITION 1. *Let* $x = \{\xi_k\} \in l_p$, $p \geq 1$, $\xi_k \neq 0$ *for every* k; *let* $\{s_n\}$ *be a sequence of complex numbers* ($s_n \to \infty$ as $n \to \infty$) *lying in the region* $\Re s > 0$, $|\arg s| \leq \phi < \pi/2$, *and let* $x_n = \{\xi_k e^{-\lambda_k s_n}\}$ $(n = 1, 2, \cdots)$, *where*

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k \to \infty \qquad (k \to \infty).$$

Then the linear manifold determined by $\{x_n\}$ *is dense in* l_p.

Proof. Let $x^* = \{\alpha_k\}$ be a linear continuous functional in l_p such that

$$x^*(x_n) = \sum_{k=1}^{\infty} \alpha_k \xi_k e^{-\lambda_k s_n} = 0 \qquad (n = 1, 2, \cdots). \tag{1}$$

Since

$$\sum_{k=1}^{\infty} |\alpha_k \xi_k| \leq \left(\sum_{k=1}^{\infty} |\xi_k|^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{\infty} |\alpha_k|^q\right)^{\frac{1}{q}} < \infty,$$

the Dirichlet series $\Sigma_{k=1}^{\infty} \alpha_k \xi_k e^{-\lambda_k s}$ is absolutely and uniformly convergent in the closed half-plane $\Re s \geq 0$. Hence it represents an analytic function

$$f(s) = \sum_{k=1}^{\infty} \alpha_k \xi_k e^{-\lambda_k s} \qquad (s = \sigma + it) \tag{2}$$

which is certainly holomorphic in the half-plane $\sigma > 0$. Furthermore, from (1) we see that the function f(s) has infinitely many zeros s_1, s_2, $\cdots$ lying in an angle

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