POWER MAPS IN RINGS

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It requires very little technique, and no knowledge of any deep results, to discuss groups in which, for a fixed integer $n > 1$, $(xy)^n = x^n y^n$ for all x and y in the group. One cannot expect the same situation to hold for rings, because certain special cases involving such an identity provide interesting theorems. For instance, the theorem of Jacobson which asserts that a ring in which $x^n = x$ for all x is commutative should be a corollary of any results obtained about a ring in which $(xy)^n = x^n y^n$.

In this paper we first study rings in which $(xy)^n = x^n y^n$; later we consider rings in which $(x + y)^n = x^n + y^n$. In the last section we assume that both relations hold. Our theorems then say that while the rings need not be commutative, they are nearly commutative in the sense that all commutators turn out to be nilpotent. The existence of nil rings in which $x^n = 0$ for all x probably rules out the possibility of any stronger result.

1. RINGS IN WHICH $(xy)^n = x^n y^n$

In this section we assume that R is a ring in which $(xy)^n = x^n y^n$ for all $x, y \in R$ and a fixed integer $n > 1$. We prove

THEOREM 1. *Let* R *be a ring in which* $(xy)^n = x^n y^n$ *for all* $x, y \in R$ *and a fixed integer* $n > 1$; *then every commutator* $ab - ba$ *in* R *is nilpotent. Moreover, the nilpotent elements of* R *form an ideal.*

Proof. Using the Jacobson structure theory and settling the theorem first for division rings, we shall ascend to general rings.

So suppose that R is a division ring. Since $(xy)^n = x^n y^n$, cancelling an x on the left and a y on the right in this identity we see that $(yx)^{n-1} = x^{n-1} y^{n-1}$ for all $x, y \in R$. Hence

$$y^n x^n = (yx)^n = (yx)(yx)^{n-1} = (yx)(x^{n-1} y^{n-1}) = yx^n y^{n-1}.$$

This leads us to $y^{n-1} x^n = x^n y^{n-1}$ for all $x, y \in R$. Let D be the subdivision ring of R generated by all the x^n. D is invariant under all inner automorphisms of R, so that by the Brauer-Cartan-Hua theorem [2] either $D = R$ or D is contained in Z, the center of R. If $D \subset Z$, then, since $x^n \in D \subset Z$ for all $x \in R$, it follows from a theorem of Kaplansky [5] that R is a commutative field. On the other hand, if $D = R$, then since $y^{n-1} x^n = x^n y^{n-1}$ for all $x, y \in R$, y^{n-1} commutes with the generators of D, hence with every element of D, and therefore with every element of R. Thus $y^{n-1} \in Z$ for all $y \in R$; again by Kaplansky's theorem we can conclude that R is commutative. Thus a division ring satisfying $(xy)^n = x^n y^n$ must be a field.

Suppose now that R is a primitive ring; if it should happen that R is not a division ring, then the 2-by-2 matrices over some division ring would be a homomorphic

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