CLOSED METRIC FOLIATIONS

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1. We adopt here the notation and definitions of [5]: M is a C^∞ n-dimensional manifold with a p-dimensional foliation F and a fibrelike Riemannian ds^2. (In view of the results of [3], it seems wise to reserve the term "bundle-like" for the case where the leaves are totally geodesic.) We shall be concerned solely with the case where all leaves are closed subsets of M. The properties mentioned so far will be indicated by saying that M has a *closed metric foliation.* The quotient space $B = M/F$ is the space formed from M by identifying each leaf to a point, and $\pi: M \to B$ is the identification map. If L is a leaf, H(L) is the holonomy group of L.

The concept of V-*manifold* has been defined by Satake [6]; roughly speaking, a V-manifold is a connected Hausdorff space which is locally homeomorphic to the quotient of Euclidean space by a finite, differentiable transformation group. For precise definitions of V-manifold and V-bundle, we refer to Baily [1]. We also introduce the notion of V-*fibre space*. defined by dropping the structural group from the definition of V-bundle. We shall use $\{U, G, \phi\}$ to denote a local uniformizing structure on an open set in the V-manifold B, λ to denote an injection $\{U, G, \phi\} \to \{U', G', \phi'\}$, and h_U to denote an anti-isomorphism of G into a group of fibre mappings of a fibre space B_U over U onto itself. All these objects are assumed to have the properties postulated in [1].

In a previous paper [5], we described the structure of a metric foliation in the neighborhood of a leaf. The description is incorrect for a leaf which is not a closed set [2], but is valid in the case in which we are now interested. (The difficulty lies in the fact that the construction required for the theorem may not be possible in M, but must be carried out in an auxiliary space. Corollary 1 in [5] remains correct in the general case.) Our present aim is to discuss the structure globally.

THEOREM. *A closed metric foliation* F *of a complete Riemannian manifold* M *is a* V-*fibre space over a* V-*manifold* B *as a base space, where* $B = M/F$ *and* $\pi: M \to B$ *is the identification map.*

2. We need two lemmas.

LEMMA 1. *For a closed metric foliation, the holonomy group of each leaf is finite.*

Proof. H(L) may be considered as a group of isometries of the sphere of unit vectors orthogonal to the leaf L at some arbitrary point of L. By the exponential map, this sphere may be embedded in a small (n - p)-plane formed by orthogonal geodesics radiating from this point. The orbit of a point P under H(L) is just the intersection of the leaf through P with this embedded sphere, hence is a closed subset of the sphere. We first show that this orbit is a finite set. Suppose not; then it has a point of accumulation, which belongs to the orbit since it is a closed set. But

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