ON POLYHEDRA IN SPACES OF CONSTANT CURVATURE

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An elementary procedure for proving the formula

$$F = \alpha + \beta + \gamma - \pi, \tag{1}$$

where F is the area of a geodesic triangle on the unit sphere and where α, β, γ are the angles, leads to analogous relations for polyhedra in spaces of constant positive curvature of an even number of dimensions. We shall here confine ourselves to the case of the four-dimensional spherical space $\mathfrak{S}_4$ of curvature +1.

1. PENTAHEDRA

We shall first derive an analogue to (1) which expresses the volume of a pentahedron in terms of the solid angles at its vertices and the angles formed by the lateral surfaces.

The five lateral surfaces are tetrahedra in three-dimensional spaces of curvature +1. Each of these spaces divides $\mathfrak{S}_4$ into two congruent semispaces which shall be denoted by $1, \bar{1}$; $2, \bar{2}$; $3, \bar{3}$; $4, \bar{4}$; $5, \bar{5}$, corresponding to the five lateral surfaces S_1, S_2, S_3, S_4, S_5. The set of the interior points of the pentahedron can be assumed to be the intersection of the semispaces 1, 2, 3, 4, 5. Let V be the volume of the pentahedron, and let (1, 2, 3, 4, 5) be the volume of the intersection of 1, 2, 3, 4, 5. We then have

$$V = (1, 2, 3, 4, 5). \tag{2}$$

It is obvious that this notation immediately furnishes the relations

$$(1, 2, 3, 4, 5) + (1, 2, 3, 4, \bar{5}) = (1, 2, 3, 4), \tag{3}$$

where (1, 2, 3, 4) is the intersection of the spaces 1, 2, 3, 4. Taking into account that

$$(1, 2, 3, 4, 5) = (\bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}), \tag{4}$$

we obtain the formulae

$$5V + \sum (1, 2, 3, 4, \bar{5}) = \sum (1, 2, 3, 4), \tag{5a}$$

$$4\sum (1, 2, 3, 4, \bar{5}) + 2\sum (1, 2, 3, \bar{4}, \bar{5}) = \sum (1, 2, 3, \bar{5}), \tag{5b}$$

$$V + \sum (1, 2, 3, 4, \bar{5}) + \sum (1, 2, 3, \bar{4}, \bar{5}) = \frac{4\pi^2}{3}, \tag{5c}$$

where the summations extend over all possible distributions of the bars among the numbers 1, 2, 3, 4, 5.

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