ON THE HERMITEAN PRODUCT OF ORDERED POINT SETS ON THE UNIT CIRCLE

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1. INTRODUCTION

The following proposition was suggested to us by V. L. Klee, who described it as a conjecture of C. J. Titus and J. L. Ullman.

(1.1) *If*

$$0 \le \phi_j < 2\pi \quad \textit{for } j = 1, 2, \cdots, n,$$

$$0 \le \psi_j < 2\pi \quad \textit{for } j = 1, 2, \cdots, n,$$

$$\phi_j < \phi_{j+1},\ \psi_j < \psi_{j+1} \quad \textit{for } j = 1, \cdots, n-1, \textit{ and}$$

$$\sum_{j=1}^{n} e^{i\phi_j} = \sum_{j=1}^{n} e^{i\psi_j} = 0,$$

then

$$\sum_{j=1}^{n} e^{i(\phi_j - \psi_j)} \neq 0.$$

In Section 2, we prove the conjecture, reducing the problem to the case where $\phi_j \ge \psi_j$ for $j = 1, 2, \cdots, n$. That this can be done was suggested to us by N. H. Kuiper, who proved it independently. In Section 3, we consider the continuous analogue of the problem. In Section 4, we see that if $n \equiv 0 \pmod 4$, the sum considered in Section 1 has 0 as g.l.b., and that the g.l.b. of the sum is positive if $n \not\equiv 0 \pmod 4$. We compute a lower bound for the sum, but do not prove that the value found really is the g.l.b. of $|\Sigma e^{i(\phi_j - \psi_j)}|$.

2. PROOF OF THE CONJECTURE

We first prove the following theorem.

(2.1) *If* $\quad 0 = \phi_1 < \phi_2 < \cdots < \phi_n < 2\pi, \qquad 0 = \psi_1 < \psi_2 < \cdots < \psi_n < 2\pi,$

$$\phi_j \ge \psi_j \text{ for } j = 1, 2, \cdots, n, \textit{ and}$$

$$\sum_{j=1}^{n} e^{i\phi_j} = \sum_{j=1}^{n} e^{i\psi_j} = 0,$$

then

$$\sum_{j=1}^{n} e^{i(\phi_j - \psi_j)} \neq 0.$$

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