ON GROUP ALGEBRAS OF p-GROUPS

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1. INTRODUCTION

Let G be group, and K a field of characteristic $p \neq 0$. The group algebra Γ_G of G over K consists of all formal sums $\Sigma\, \alpha(g)\, g$, where $g \in G$, $\alpha(g) \in K$, and $\alpha(g) = 0$ for all but finitely many g. The operations + and $\cdot$ are defined in the natural way. Denote by Δ_G the set of all those sums $\Sigma\, \alpha(g)\, g$ for which $\Sigma\, \alpha(g) = 0$; Δ_G is an ideal of Γ_G, generally called the fundamental ideal. Jennings [2] and Lombardo-Radice [3] have both shown that Δ_G is nilpotent if G is a finite p-group. In this paper, we intend to show that the converse is also true. These results will then be applied to the case where G is a locally finite p-group.

The situation where the fundamental sequence $\Delta_G \supseteq \Delta_G^2 \supseteq \Delta_G^3 \supseteq \cdots$ terminates in a finite number of steps at an ideal different from zero appears to be more difficult to analyze. Here, we shall only consider the case where G has exponent p and K is Z_p, the ring of integers modulo p.

2. NILPOTENCE OF THE FUNDAMENTAL IDEAL

LEMMA 2.1. *The elements* $g - 1$ *for all* $g \neq 1$ *in* G *are a basis for* Δ_G. *If* $(h_i)_{i \in I}$ *is a set of generators for* G, *then the subalgebra of* Γ_G *generated by the elements* $h_i^{\pm 1} - 1$ *is exactly* Δ_G. *In fact, the left ideal of* Γ_G *generated by the elements* $h_i - 1$ *is* Δ_G.

Proof. If $\Sigma\, \alpha(g)\, g \in \Delta_G$, then $\Sigma\, \alpha(g) = 0$ and therefore

$$\sum \alpha(g)\, g = \sum \alpha(g)\, g - \sum \alpha(g) = \sum \alpha(g)(g - 1).$$

Hence, the elements $g - 1$ span Δ_G. It is clear that they are linearly independent.

Let $(h_i)_{i \in I}$ be a set of generators for G, and let J be the subalgebra generated by all $h_i^{\pm 1} - 1$. Clearly, $J \subseteq \Delta_G$. If $g \in G$, then

$$g - 1 = h_{i(1)}^{\varepsilon(1)} \cdots h_{i(k)}^{\varepsilon(k)} - 1,$$

where $\varepsilon(j) = \pm 1$. Applying the identity

$$XY - 1 = (X - 1) + (Y - 1) + (X - 1)(Y - 1)$$

to the right-hand side of this equation sufficiently often, we obtain a representation of $g - 1$ as a linear combination of products of the $h_i^{\varepsilon} - 1$. Hence $g - 1$ is in J, and hence $\Delta_G \subseteq J$. Therefore, $J = \Delta_G$.

Let Λ be the left ideal generated by the elements $h_i - 1$. Then

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