GROUP COMMUTATORS OF BOUNDED OPERATORS IN HILBERT SPACE

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1. In the sequel only bounded (linear) operators A, B, $\cdots$ on a Hilbert space of elements x, y, $\cdots$ will be considered. For any such operator A, let $W = W(A)$ denote the closure of the set of complex numbers (Ax, x) with $\|x\| = (x, x)^{1/2} = 1$. It is known that W is a bounded convex set containing sp(A), the spectrum of A, and that in case A is normal, W is the least closed convex set containing sp(A) (Hausdorff, Toeplitz); see, for example [4], pp. 34 ff. An operator A will be called *nonsingular (invertible)* if it possesses a unique right (hence, a unique left) bounded inverse A^{-1}. In case A and B are nonsingular, let $D = ABA^{-1}B^{-1}$, the group commutator of A and B. It will be supposed throughout this paper that A commutes with D, so that

$$(1) \qquad AD = DA, \qquad D = ABA^{-1}B^{-1}.$$

It is known that if, in addition to (1), A and B are finite-dimensional unitary matrices and if the spectrum of B is contained in some open semicircle on the circle $|z| = 1$, then necessarily $D = I$, that is, $AB = BA$; see [2, Theorem 197], also [3]. In the present paper various generalizations of this result will be obtained; in particular it will be shown that the restriction that A and B be finite matrices can be removed. Since, when B is unitary, the above assumption concerning sp(B) is equivalent to the condition that 0 fails to belong to the set W(B), it is clear that the earlier assertion for the case where A and B are finite-dimensional and unitary is contained in

(I) *Let* A *and* B *be unitary (so that* $D = ABA^{-1}B^{-1}$ *is unitary) and satisfy* (1). *Then either* $AB = BA$ *or* 0 *belongs to the set* W(B).

If N is any nonsingular normal operator, it is easy to see that 0 belongs to the set W(N) if and only if 0 belongs to the set $W(N^{-1})$. Consequently, (I) is seen to be a consequence of the more general result

(II) *Let* A *be unitary, and let* B *be an arbitrary nonsingular operator satisfying* (1). *Then at least one of the following cases must hold:* (i) $sp(D) = 1$ *only, or* (ii) 0 *belongs to* W(B), *or* (iii) 0 *belongs to* $W(B^{-1})$.

2. *Proof of* (II). Suppose that z belongs to sp(D); then, as $m \to \infty$,

$$(2) \qquad \text{either} \quad (D - z)x_m \to 0 \quad \text{or} \quad (D^* - \bar{z})x_m \to 0,$$

for some sequence of elements x_m satisfying $\|x_m\| = 1$. It will be shown that if $z \neq 1$, that is, if (i) fails to hold, condition (2) implies that either (ii) or (iii) of (II) must hold.

It is seen from $DB = ABA^{-1}$ and an application of (1) that $D^2B = A^2BA^{-2}$ and, in general, that

Received July 2, 1960.

This research was supported by the United States Air Force through the Air Force Office of Scientific Research of the Air Research and Development Command, under Contract No. AF 18 (603) - 139. Reproduction in whole or in part is permitted for any purpose of the United States.