THE TOPOLOGICAL STRUCTURE OF TRAJECTORIES

John Gary

1. INTRODUCTION

Let V_n be a connected differentiable manifold of class C^r $(r \geq 1)$, and let $X(x, t)$ $(x \in V_n,\ t \in E_1)$ define a field of tangent vectors on $V_n \times E_1$. We shall always assume that $X(x, t)$ is of class C^r $(r \geq 1)$. Thus we have a system of equations,

$$\frac{dx}{dt} = X(x, t).$$

If $x(t)$ is a solution of these equations, we call the curve $(x(t), t) \subset V_n \times E_1$ a *trajectory*. We consider only maximal trajectories; that is, the function $x(t)$ is defined for $a < t < b$, where the interval (a, b) is maximal (a or b may be infinite).

Lefschetz has defined the general solution of a system of equations [1, p. 39]. We shall give a second definition and analyze the relation of these general solutions to the topological structure of the space of trajectories.

2. DIFFERENTIAL SYSTEMS WHICH ADMIT A SECTION

Definition 2.1. A *section* for a differential system on $V_n \times E_1$ is a manifold V_n', together with a homeomorphism f of V_n' into $V_n \times E_1$, with the property that $f(V_n')$ meets each trajectory exactly once.

THEOREM 2.2. *Assume we are given a differential system defined by a function* $X(x, t)$ *on* $V_n \times E_1$. *The trajectories of this system form a decomposition of* $V_n \times E_1$ *into closed sets. Let* V *be the space obtained by the identification of each trajectory to a single point. If* V *is a Hausdorff space, then* V *is an* n*-dimensional manifold. If* $X(x, t)$ *and* V_n *are of class* C^m, *then* V *is of class* C^m $(m \geq 1)$.

Proof. Let $p\colon V_n \times E_1 \to V$ be the mapping induced by the identification. Let $\{N_i\}$ be a countable collection of n-cells which form a basis for the topology of V_n, and $\{I_j\}$ a collection of 1-cells which form a basis for E_1. Then the collection $\{N_i \times I_j\}$ is a basis for $V_n \times E_1$. Choose $t_j \in I_j$ and let $f_{ij}\colon N_i \to N_i \times t_j$ be defined by $f_{ij}(x) = (x, t_j)$. Let $g_{ij}\colon N_i \to V$ be defined by $g_{ij} = p(f_{ij}(x))$, and define $U_{ij} = g_{ij}(N_i)$. Then $p^{-1}(U_{ij})$ is the set of all points which lie on a trajectory passing through $N_i \times t_j$. Thus $p^{-1}(U_{ij})$ is open in $V_n \times E_1$, and U_{ij} is open in V [1, p. 52]. If U is an open set in V and $y \in U$, then there exists a cell $N_i \times I_j$, contained in $p^{-1}(U)$, with the property that $p^{-1}(y)$ meets $N_i \times t_j$. Then $y \in U_{ij}$ and $U_{ij} \subset U$. Thus, the collection $\{U_{ij}\}$ is a basis for the topology of V. The mappings g_{ij} are one-to-one, and the g_{ij}^{-1} are continuous, since the g_{ij} are open.

Now assume that $X(x, t)$ is of class C^m. Then the solution $x(t, x_0, t_0)$, where $x(t_0, x_0, t_0) = x_0$, is of class C^m in (x_0, t_0). We must show that

$$g_{pq}^{-1} g_{ij}\colon g_{ij}^{-1}(U_{ij} \cap U_{pq}) \to g_{pq}^{-1}(U_{ij} \cap U_{pq})$$

Received October 7, 1959.