A CHARACTERIZATION OF EUCLIDEAN n-SPACE

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The n-sphere S^n has the property that, corresponding to each neighborhood U of a point p in S^n, there exists a homeomorphism h of S^n *onto* itself such that $h(S^n - U)$ lies in U. Of course, h may be taken to be an inversion of S^n in an (n - 1)-sphere; but the interesting observation is that S^n is the only n-manifold with this property. The following result is then a simple characterization of the n-sphere.

THEOREM 1. *Let* M *be an* n*-manifold, and suppose there is a point* $p \in M$ *such that, for each neighborhood* U *of* p, *there exists a homeomorphism* h *of* M *onto itself such that* $h(M - U) \subset U$. *Then* M *is an* n*-sphere.*

Proof. First we note that M need not be assumed to be compact. For if U is chosen to have compact closure in M, then the corresponding homeomorphism h of M onto itself carries the closed set M - U into the compact set $\overline{U}$, whence h(M - U) is compact and M is the union of the compact sets $\overline{U}$ and M - U.

Now let C be a closed n-cell in M with $p \in C^o$, the interior of C, and suppose that C has been selected so that its boundary βC is a parameter (n - 1)-sphere for some value of t $(0 < t < 1)$, in a homeomorphism of $S^{n-1} \times I^1$ into M. Letting $\overline{M - C} = D$, we have $M - \beta C = C^o \cup D^o$. By hypothesis, there exists a homeomorphism h of M onto itself such that $h(D) \subset C^o$. Clearly, $h(\beta C) \subset C^o$ and

$$M - h(\beta C) = h(C^o) \cup h(D^o).$$

Also we know that $C - h(\beta C) = h(D^o) \cup B$, where $B = C - h(D)$.

There exists an imbedding g of C into the n-sphere S^n. The (n - 1)-sphere $gh(\beta C)$ is a parameter sphere in S^n, as described above. Hence by Brown's Theorem 5 in [1], $gh(\beta C)$ bounds two n-cells in S^n. Obviously, gh(D) is one of these n-cells.

Thus M is the union of two n-cells C and D meeting on their common boundary, and hence M is an n-sphere.

The same sort of inversion property also may be expected to characterize Euclidean n-space E^n, and this is easily shown to be true.

THEOREM 2. *Let* M *be a noncompact* n*-manifold, and suppose there is a point* $p \in M$ *such that, for each neighborhood* U *of* p, *there exists a homeomorphism* h *of* M - p *onto itself such that* $h(M - U) \subset U$. *Then* M *is homeomorphic to* E^n.

Proof. As in the proof of Theorem 1, let C be a closed n-cell in M such that βC is a parameter (n - 1)-sphere and $p \in C^o$. Again setting $\overline{M - C} = D$, we have $M - \beta C = C^o \cup D^o$, and there exists a homeomorphism h of M - p onto itself such that $h(D) \subset C^o - p$. As before, $h(\beta C)$ separates C, and we claim that p and $h(D^o)$ lie in the same component of $C - h(\beta C)$. For if not, then h(D) would be a closed n-cell, and hence M would be an n-sphere, which contradicts the assumption that M is noncompact.

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