DIFFERENTIABLE ISOTOPIES ON THE 2-SPHERE

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Let Diff S^n denote the group of diffeomorphisms of degree +1 and of class C^r $(1 \leq r \leq \infty)$ carrying the unit n-sphere onto itself, topologized by requiring closeness of the maps and their partial derivatives through order r. A path in this space is called a *regular isotopy;* it is a map of the reals R into the space which is constant on the set $t \leq 0$ and on the set $t \geq 1$. We say a path is *differentiable* if the induced map of $S^n \times R$ onto S^n is of class C^1; a differentiable path is also called a *differentiable isotopy.* If two maps are regularly isotopic, they are differentiably isotopic as well [2, Lemma 1.6].

The group Γ^{n+1} of Milnor and Thom is defined as a quotient group of the group $\pi_0(\text{Diff } S^n)$ of path components of this space. The object of the present paper is to give an elementary proof that $\pi_0(\text{Diff } S^2)$, and hence Γ^3, vanishes. This result has since been generalized by Smale [3]. Interest in the group Γ^{n+1} stems from its close connection with the existence of distinct differentiable structures on manifolds; the fact that $\Gamma^3 = 0$, for example, implies the uniqueness theorem for differentiable structures on 3-manifolds [2]. The group $\pi_0(\text{Diff } S^n)$ does not in fact depend on the choice of r; for simplicity we shall prove our result only in the case r = 1.

1. Let $\mathcal{E}^r(R^m, R^n)$ $(m \leq n)$ denote the space of those embeddings of class C^r of R^m in R^n which equal the inclusion map i outside some compact subset of R^m; it is topologized as was Diff S^n. (The map i is defined by the equation

$$i(x_1, x_2, \cdots, x_m) = (x_1, \cdots, x_m, 0, \cdots, 0).)$$

A *loop* in this space will be assumed to be based at i and to be constant for t near 0 and for t near 1. Every element ϕ of Diff S^n is regularly isotopic to an element ϕ_1 which equals the identity in a neighborhood of the north pole [2, Lemma 8.1]; stereographic projection carries ϕ_1 into an element f of $\mathcal{E}^r(R^n, R^n)$. A path in $\mathcal{E}^r(R^n, R^n)$ is carried by the inverse of this projection into a path in Diff S^n; hence our problem reduces to showing that $\pi_0(\mathcal{E}^1(R^2, R^2)) = 0$.

1.1. LEMMA. *Let* f_t *be a differentiable loop in* $\mathcal{E}^2(R^1, R^2)$ *which is homotopic to a loop in the subspace* $\mathcal{E}^2(R^1, R^1)$, *the homotopy passing through differentiable loops. Then there exists a differentiable loop* G_t *in* $\mathcal{E}^1(R^2, R^2)$ *such that for each* t, $G_t f_t$ *maps* R^1 *into itself.*

Proof. Given $g \in \mathcal{E}^2(R^1, R^2)$, there exists a neighborhood of g such that if h lies in this neighborhood, there exists a C^1 diffeomorphism J of R^2 which carries the set $g(R^1)$ onto $h(R^1)$. The diffeomorphism is obtained by sliding $g(R^1)$ along its normal lines onto $h(R^1)$ and leaving everything fixed outside a neighborhood of these sets. Because the maps are of class C^2, there is no difficulty in carrying out this construction; details are left to the reader. Similarly, if h_t is a differentiable loop approximating closely enough the differentiable loop g_t, then the diffeomorphism J_t may be chosen to be a differentiable loop in $\mathcal{E}^1(R^2, R^2)$.

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