SOME REMARKS ON SET THEORY, VIII

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This paper discusses some problems similar to questions considered in earlier communications of the same title [2], [3] and to some questions treated by P. Erdös and R. Rado [4], [5].

1. ON INDEPENDENT SETS

Let M be a set (in this note, M will denote the set of real numbers), and to each $x \in M$, let there correspond a set $S(x) \subset M$, called the *picture* of s, such that $x \notin S(x)$. A subset M' of M is called *independent* (or *free*) if, for each pair of points x and y in M, $x \notin S(y)$ and $y \notin S(x)$. In [2, I, p. 52] it was conjectured that if M is the set of real numbers, and if the measure of S(x) is bounded and S(x) is not everywhere dense, then there always exists an independent pair. In fact, it is easy to see that if we assume $\mathfrak{c} = \aleph_1$, then this conjecture is false. To construct a counterexample, we well-order M into an Ω_1-sequence $\{x_\alpha\}$ $(\alpha < \Omega_1)$. For each α, we write

$$S(x_\alpha) = S_1(x_\alpha) \cup S_2(x_\alpha),$$

where $S_1(x_\alpha)$ is the interval $(x_\alpha, x_\alpha + 1)$, and where $x_\beta \in S_2(x_\alpha)$ provided $\beta < \alpha$ and x_β does not lie in the interval $(x_\alpha - 1, x_\alpha)$. Clearly, $S(x_\alpha)$ has measure 1 (the set $S_2(x_\alpha)$ is at most denumerable) and is not everywhere dense, and no two points are independent.

Instead of the hypothesis that $\mathfrak{c} = \aleph_1$, we have used only the hypothesis that the measure of every set of power less than $\mathfrak{c}$ is 0. In fact, we need only the hypothesis that the set of real numbers can be well-ordered into a sequence $\{x_\alpha\}$ $(\alpha < \Omega_\mathfrak{c})$ such that every set which is not cofinal with $\Omega_\mathfrak{c}$ has measure 0. Denote this hypothesis by H_0. We do not know whether H_0 is equivalent to the hypothesis that each set of power less than $\mathfrak{c}$ has measure 0. Further, we do not know whether, if S(x) has the properties above, the negation of H_0 implies the existence of an independent pair.

Piranian (private communication) recently asked what can be said about independent points if each S(x) has measure 0 and is not everywhere dense.

THEOREM 1. *If* S(x) *has measure* 0 *and is not everywhere dense, there exists an independent pair; under the additional assumption* H_0, *an independent triplet need not exist.*

Proof. Let $A = \{a_n\}$ $(1 \leq n < \infty)$ be a denumerable dense set. Then $\bigcup_{n=1}^{\infty} S(a_n)$ is clearly of measure 0, and its complement contains a point b. Since S(b) is not everywhere dense, there exists an m such that $a_m \notin S(b)$. Clearly, a_m and b are independent.

On the other hand, let $\{x_\alpha\}$ $(\alpha < \Omega_\mathfrak{c})$ be a well-ordering of M. For $0 < \alpha < \Omega_\mathfrak{c}$, let $S(x_\alpha)$ be the set of those x_β $(\beta < \alpha)$ that have the same sign as x_α (here the sign

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