ON SEQUENCES OF SUBORDINATE FUNCTIONS

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Let f(z) and g(z) be two functions regular in the disk $|z| < 1$. If there exists a function $\phi(z)$ that is regular in $|z| < 1$ and satisfies $|\phi(z)| < 1$ and $\phi(0) = 0$, such that

$$g(z) = f(\phi(z))$$

in $|z| < 1$, then g(z) is called *subordinate* to f(z) (see for instance [2, p. 163]). The condition implies that $g(0) = f(0)$ and $|g'(0)| \leq |f'(0)|$. The relation of subordination is transitive. We shall prove the following theorems:

THEOREM 1. *Let the functions* $f_n(z)$ *be regular in* $|z| < 1$, *let* $\alpha_n = f_n'(0)$ *be positive, and let* $f_n(z)$ *be subordinate to* $f_{n+1}(z)$. *Then the condition*

$$\alpha = \lim_{n\to\infty} \alpha_n < \infty$$

is necessary and sufficient in order that $\{f_n(z)\}$ *converges uniformly in* $|z| \leq r$ *for every* $r < 1$.

THEOREM 2. *Let the functions* $f_n(z)$ *be regular in* $|z| < 1$, *let* $\alpha_n = f_n'(0)$ *be positive, and let* $f_{n+1}(z)$ *be subordinate to* $f_n(z)$. *Then the sequence* $\{f_n(z)\}$ *converges uniformly in* $|z| \leq r$ *for every* $r < 1$. *The limit function is constant if and only if*

$$\alpha = \lim_{n\to\infty} \alpha_n = 0.$$

Remarks. 1. Note the difference in the assumptions: In Theorem 1 we assume that $f_n(z)$ is subordinate to $f_{n+1}(z)$, whereas in Theorem 2 we assume the reverse relationship.

2. In Theorem 1 we have $\alpha_{n+1} \geq \alpha_n$. Therefore either the limit α exists or $\alpha_n \to \infty$. In Theorem 2 we have $\alpha_{n+1} \leq \alpha_n$. Hence the limit α always exists and is nonnegative.

3. In Theorem 2, it is of course essential to assume that $f_n'(0)$ is real and nonnegative, as the example $f_n(z) = (-1)^n z$ shows.

4. Theorem 1 implies that if its hypothesis is satisfied and if $|f_n(z)| \leq K$ in some neighborhood of $z = 0$, then $|f_n(z)| \leq M(r)$ in $|z| \leq r$ for every $r < 1$. The functions $f_n(z) = e^{nz}$ give an example for Theorem 1 with $\alpha_n = n \to \infty$.

We need two lemmas. We denote by A the class of all functions $\phi(z)$ that are regular in $|z| < 1$ and satisfy the conditions $|\phi(z)| < 1$ and $\phi(0) = 0$.

LEMMA 1. *Every function* $\zeta = \phi(z)$ *of class* A *with* $\phi'(0) \geq \sigma > 0$ *maps the disk*

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