HOMOLOGY THEORY FOR LOCALLY COMPACT SPACES

A. Borel and J. C. Moore

In this paper, we develop a homology theory for locally compact spaces. On compact metric spaces, our theory is equivalent to the Steenrod homology theory [8]. The main purpose of introducing it is to obtain a Poincaré duality theorem for cohomology manifolds (see Section 7). The subject matter of the present paper is essentially the same as that of [3, Chapter II]. However, more emphasis has been put on the homology theory, which is treated from a slightly different point of view. Cohomology manifolds, which were the main concern of [3, Chap. I, II], will here be discussed more briefly.

The notation will in general be that of [7]. We assume familiarity with sheaf theory [4], [7]. In particular, the following concepts and notation will be used. A *family of supports* Φ in the space X is a collection of closed subsets such that

i) if $A, B \in \Phi$, then $A \cup B \in \Phi$, and

ii) if $B \in \Phi$ and A is a closed subset of B, then $A \in \Phi$.

Given a sheaf $\mathscr{S}$ on X, the symbols $\Gamma(\mathscr{S})$, $\Gamma_\Phi(\mathscr{S})$, $\mathscr{S}(A)$ will denote respectively the sections of $\mathscr{S}$, the sections of $\mathscr{S}$ with support in Φ, and the sections of $\mathscr{S}$ over the subspace A of X. Note that by *section* we shall always mean continuous section.

If A is a subspace of X and $\mathscr{S}$ is a sheaf on X, we denote by $\mathscr{S}|A$ the restriction of $\mathscr{S}$ to A. Further, if A is locally closed, we denote by $\mathscr{S}_A$ the sheaf on X which induces $\mathscr{S}|A$ on A and zero on X - A. Recall that if A is open in X, the sequence

$$0 \to \mathscr{S}_A \to \mathscr{S} \to \mathscr{S}_{X-A} \to 0$$

is an exact sequence of sheaves on X.

A sheaf $\mathscr{S}$ on X is *flabby* is the restriction map $\Gamma(\mathscr{S}) \to \mathscr{S}(A)$ is surjective for every open subspace A of X; it is called *soft* if $\Gamma(\mathscr{S}) \to \mathscr{S}(A)$ is surjective for every closed subspace A of X. If Φ is a family of supports in X, then $\mathscr{S}$ is *soft relative to* Φ if $\mathscr{S}|A$ is soft for every $A \in \Phi$. When the family Φ is paracompactifying [7], this is equivalent to saying that $\Gamma_\Phi(\mathscr{S}) \to \Gamma_\Phi(\mathscr{S}|A) = \mathscr{S}(A)$ is surjective for every $A \in \Phi$. Thus the word "flabby" is the translation of *flasque*, and "soft" the translation of *mou.*

Throughout this paper, we shall assume that K is a Dedekind ring given once and for all. All modules are assumed to be K-modules, and $\otimes$, Tor, Hom, and Ext are taken over K. All sheaves in addition to their stated properties will be assumed to be sheaves of K-modules. Finally, all topological spaces will be assumed to be Hausdorff and, from Section 2 on, locally compact.

Received November 12, 1959.

The second author was partially supported by the Air Force Office of Scientific Research under Contract AF 18(600), during the period in which this work was done.