PARTITIONS INTO PRIME POWERS

E. Grosswald

1. INTRODUCTION

Let p(n, m; k) stand for the number of partitions of the integer n into k-th powers of primes p ($2 \leq p \leq m$). Clearly, it is sufficient to consider only $m \leq n^{1/k}$. If $m \geq n^{1/k}$, or if k = 1, mention of the corresponding parameter is usually omitted, and we write simply p(n; k), or p(n, m); the total number of partitions of n into primes is denoted by p(n). Hardy and Ramanujan [4] proved that

$$\log p(n, k) \sim (k+1)\left\{\Gamma\left(2+\frac{1}{k}\right)\zeta\left(1+\frac{1}{k}\right)\right\}^{k/(k+1)}\left(\frac{n}{\log^k n}\right)^{1/(k+1)},$$

so that, in particular,

$$\log p(n) \sim 2\pi\left(\frac{n}{3 \log n}\right)^{1/2}.$$

Brigham [2] obtained an asymptotic formula for a certain weighted partition function. In 1953, Haselgrove and Temperley [5] obtained an asymptotic formula for partitions into parts (with or without restriction on their number) selected from some preassigned set A of integers, provided that A satisfies certain conditions. The results obtained are of considerable generality, because most sets of interest satisfy the required conditions almost trivially; the primes, however, are a borderline case. The formulae still hold, but the corresponding justification is far from simple, and it is presented (as are some other points of the paper) rather sketchily. Actually, Haselgrove and Temperley's formula is valid even for the partitions into prime powers p(n; k), but that is neither justified, nor even claimed in the paper. This may account for the fact that Mitsui, who in 1957 obtained [8] an asymptotic formula for p(n, m; k), credits Haselgrove and Temperley only with the determination of p(n, n; 1) instead of p(n, n; k). The more general formulae of Haselgrove and Temperley and of Mitsui are not directly comparable, because the former's restrictions refer to the number of parts, while the latter's refer to the size of the summands admitted. While preceding papers make use of the theory of functions of complex variables, Bateman and Erdös [1] use a rather elementary approach, in order to prove that p(n, m) is an increasing function of both of its arguments.

2. PURPOSE OF THE PAPER

Previously mentioned results concerning p(n, m; k) do not actually lead to asymptotic formulae in n, in the customary sense. If one replaces the parameters occurring in [5] or [8] by their asymptotic values as functions of n, m and k, one obtains (see [8], Corollaries 1 and 2) results of the form

$$p(n, m; k) = \tilde{p}(n, m; k) \exp\{O(n^{1/(k+1)} \log^c n)\},$$

Received November 19, 1959.

This paper was written while the author held a National Science Foundation Fellowship.