THE AUTOMORPHISM GROUP OF THE FREE GROUP WITH TWO GENERATORS

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Let F be the free group generated by a and b, and let F' denote the derived group [F, F] of F. The main purpose of this note is to prove

THEOREM 1. *An automorphism* G *of* F *is an inner automorphism if* $G(a) \equiv a$, $G(b) \equiv b \pmod{F'}$.

As an immediate consequence of Theorem 1, we have

THEOREM 2. *Let* A *and* I *be the automorphism group and the inner automorphism group of* F, *respectively. Then the group* A/I *is isomorphic to the group of two-by-two matrices with integer coefficients and with determinants* ± 1.

Proof of Theorem 1. It is known [2], [3] that A is generated by the three automorphisms

$$P: a \to b,\ b \to a, \qquad Q: a \to a^{-1},\ b \to b, \qquad U: a \to ab,\ b \to b.$$

Let V be the automorphism $V: a \to a,\ b \to ba$; then we have

$$PU = VP, \qquad PU^{-1} = V^{-1}P, \tag{1}$$

$$QU \equiv U^{-1}Q \pmod{I}. \tag{2}$$

(The symbol $G_1 G_2$ denotes the automorphism G_1 followed by G_2. Thus

$$G_1 G_2: a \to G_2(G_1(a)),\ b \to G_2(G_1(b)).)$$

Using the above relations, we may write an automorphism

$$G = P^{\delta_1} Q^{\varepsilon_1} U^{\lambda_1} \cdots P^{\delta_k} Q^{\varepsilon_k} U^{\lambda_k},$$

where $\delta_1, \varepsilon_1, \cdots, \delta_k, \varepsilon_k$ are 0 or 1 and $\lambda_1, \cdots, \lambda_k$ are integers, as

$$G \equiv U^{\mu_1} V^{\nu_1} \cdots U^{\mu_j} V^{\nu_j} W \pmod{I},$$

where $\mu_1, \nu_1, \cdots, \mu_j, \nu_j$ are integers and W is a word in P and Q.

Let

$$R: a \to a^{-1},\ b \to b^{-1}, \qquad S: a \to b,\ b \to a^{-1}, \qquad T: a \to b^{-1},\ b \to ba.$$

We have

$$S^2 = R, \tag{3}$$

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