ESTIMATE OF A CERTAIN LEAST COMMON MULTIPLE

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Suppose that $N_1, N_2, \cdots$ are positive integers (not necessarily distinct) such that $\Sigma\, 1/N_i = 1$. If we impose the restriction that $N_i \leq N$ for all i, how large can $\mathrm{lcm}[N_1, N_2, \cdots]$ be?

Clearly, by choosing $N_i = N$ $(i = 1, 2, \cdots, N)$, we obtain lcm = N; and on the other hand, the inequality $\mathrm{lcm}[N_i] \leq \mathrm{lcm}[1, 2, \cdots, N] \leq N!$ always holds. If we let $\Phi(N)$ denote the maximum of this lcm, then these remarks imply that $N \leq \Phi(N) \leq N!$. This trivial inequality leaves a wide gap in our knowledge of $\Phi(N)$, and it is our purpose to narrow the gap. It is fairly easy to strengthen the inequality to

$$C_1 N^2 \leq \Phi(N) \leq e^{C_2 N},$$

for example; but this improvement is slight. Our result is as follows.

THEOREM.

$$\log \Phi(N) \sim \frac{N}{\log N}.$$

Remarks. To obtain this precision, we need the prime number theorem $\pi(x) \sim x/\log x$, and its equivalent forms,

$$\log \prod_{p \leq x} p \sim x, \qquad \log \mathrm{lcm}[1, 2, \cdots, n] \sim n.$$

Depending on the reader's taste, this may or may not be "elementary;" at any rate, our method also gives

$$\frac{C_1 N}{\log N} < \log \Phi(N) < \frac{C_2 N}{\log N},$$

using only the Tchebychev estimates of $\pi(x)$.

The proof splits into two portions:

I. If $\varepsilon > 0$ and N is large, then the conditions $N_i \leq N$ and $\Sigma\, 1/N_i = 1$ imply that

$$\mathrm{lcm}[N_i] < e^{(1+3\varepsilon)N/\log N}.$$

II. If $\varepsilon > 0$ and N is large, then there exist $N_i \leq N$ with $\Sigma\, 1/N_i = 1$ and

$$\mathrm{lcm}[N_i] > e^{(1-3\varepsilon)N/\log N}.$$

Proof of I. The N_i are given with the required properties. Let S be the set of primes p which divide some N_i and such that $p \geq (1 + 2\varepsilon)N/\log N$; and for p in S,

Received November 23, 1959.