PERIODIC AND REVERSE PERIODIC CONTINUED FRACTIONS

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An important result in the theory of periodic simple continued fractions is a theorem of Galois [1, p. 2] (see, for example [2, v. 1, p. 76]). This theorem relates the value of a periodic simple continued fraction to the value of its reverse periodic simple continued fraction. The purpose of the present paper is to obtain a generalization of the Galois theorem which is applicable to a wider class of periodic continued fractions.

The general pure K-periodic continued fraction with nonzero partial numerators is considered in Section 1. The desired theorem is stated in terms of the conjugate points of the continued fraction which, in this case, are identical with the fixed points of the associated linear fractional transformation. In Section 2 the result of Section 1 is extended to mixed K-periodic continued fractions.

1. THE PURE PERIODIC CASE

Let

$$b_0 + \frac{a_1}{b_1} + \cdots + \frac{a_{K-1}}{b_{K-1}} + \frac{a_K}{b_0} + \frac{a_1}{b_1} + \cdots \quad (a_j \neq 0) \tag{1.1}$$

be a pure K-periodic continued fraction with nonzero partial numerators. The associated linear fractional transformation is

$$T_K(x) = \frac{A_{K-1}\,x + a_K A_{K-2}}{B_{K-1}\,x + a_K B_{K-2}}, \tag{1.2}$$

where the fundamental recurrence formulas for A_n and B_n are

$$\begin{aligned} A_{n+1} &= b_{n+1} A_n + a_{n+1} A_{n-1}, \\ B_{n+1} &= b_{n+1} B_n + a_{n+1} B_{n-1} \qquad (n = 0, 1, 2, \cdots), \end{aligned} \tag{1.3}$$

with $A_{-1} = 1$, $B_{-1} = 0$, $A_0 = b_0$, $B_0 = 1$, and with

$$a_{n+1} = a_{j+1}, \qquad b_n = b_j \qquad (n \equiv j \pmod{K}). \tag{1.4}$$

The conjugate points for (1.1) are defined to be the fixed points, x_1 and x_2, of the associated linear fractional transformation (1.2). Since x_1 and x_2 are the solutions of $x = T_K(x)$, or

$$B_{K-1}\,x^2 + (a_K B_{K-2} - A_{K-1})x - a_K A_{K-2} = 0, \tag{1.5}$$

they are quadratic conjugates relative to this equation. The fact that $T_K(x)$ is nonsingular follows from the condition $a_j \neq 0$ and the determinant formula

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