TRANSFORMATION GROUPS ON A K(π, 1), I

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1. INTRODUCTION

The purpose of this note is to give some results on transformation groups and fiberings for a finite-dimensional Eilenberg-MacLane space K(π, 1), a space whose one-dimensional homotopy group is π and whose remaining homotopy groups vanish. The Eilenberg-MacLane spaces are discussed in [1], and are treated completely in [2]. We are interested here only in the most elementary facts about K(π, 1) and K(π, 2). Eilenberg and Ganea [3] have pointed out that the existence of a finite-dimensional K(π, 1) for a given group π is an intrinsic algebraic property of π.

We were led to this topic by several considerations. For one thing, a special case of transformation groups on a finite-dimensional K(π, 1) is quite classical; namely, the study of transformation groups on closed Riemann surfaces of positive genus. Theorem 5.1 contains a generalization of H. A. Schwarz' theorem that no closed Riemann surface of genus larger than 1 can admit a 1-parameter family of complex analytic transformations. The theorem of Montgomery and Samelson [5] to the effect that the only compact connected Lie group which is transitive and effective on a torus is a toral group led us to conjecture and prove that the assumption of transitivity could be dropped. Paul Smith proved that the fixed point set of a cyclic transformation group of prime order p acting on a sphere has the mod p homology groups of a lower-dimensional sphere. In Theorem 3.4 we show that the fixed point set of a cyclic transformation of prime order on a K(π, 1) also inherits the mod p homology characteristics of the K(π, 1). We feel that Smith's theorem and our Theorem 3.4 are but the two extreme cases of some general relation between the homotopy groups of a space and the cyclic transformations of prime order on that space. This is the real motive for the present note, namely, to initiate the development of extensive relations between homotopy groups and cyclic transformations.

We show that if a finite-dimensional K = K(π, 1) is fibered by a connected fiber F with base B, then F is a K(π_1(F), 1) and B is a K(π_1(B), 1). Our principal result concerns those compact manifolds that are aspherical; that is, K(π, 1)-spaces that are compact manifolds. We prove that if a compact connected Lie group acts effectively on such a manifold, then the group is a toral group; moreover, this group must act freely, and if π is abelian, there is a cross section in the large. Hence, the action might be called a product action.

The K(π, 1)-spaces are assumed to be connected, locally compact, finite-dimensional ANR's. In this note we use the Alexander-Wallace-Spanier cohomology (AWS-cohomology). We denote a discrete group by π, a topological group operating on a space by (G, X), and the natural projection of X onto the orbit space X/G by

$$p: X \to X/G.$$

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