FINITENESS OF CLASS NUMBERS OF REPRESENTATIONS OF ALGEBRAS OVER FUNCTION FIELDS

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1. THE THEOREM

The purpose of this note is to establish a function-field analogue of Zassenhaus's Theorem [4] concerning the finiteness of class numbers of representations of semi-simple algebras over number fields.

Let $\mathfrak{o}$ be an integral domain with quotient field k, and let A be a finite-dimensional k-algebra. An *$\mathfrak{o}$-order* in A is defined to be an $\mathfrak{o}$-subalgebra $\mathfrak{O}$ which is finitely generated as an $\mathfrak{o}$-submodule, and such that $k\mathfrak{O} = A$. All A-modules considered will be right unitary A-modules of finite dimension over k. The $\mathfrak{O}$-submodules of an A-module V which are finitely generated as $\mathfrak{o}$-modules are called *$\mathfrak{O}$-representation submodules.* The *class number (relative to $\mathfrak{O}$)* of V is defined to be the number of nonisomorphic $\mathfrak{O}$-representation submodules M of V which *generate* V in the sense that kM = V.

If $\mathfrak{o}$ is a principal ideal domain, $\mathfrak{O}$-representation submodules have free $\mathfrak{o}$-module bases, and the definition of class number can be formulated in the obvious way in terms of matrix representations.

The $\mathfrak{O}$-representation submodules M of A generating A are the so-called *right $\mathfrak{O}$-ideals* of A. Two such $\mathfrak{O}$-ideals M and N are isomorphic if and only if there is a unit x of A such that xM = N, that is, if and only if M and N are *equivalent* $\mathfrak{O}$-ideals (see [4]). Hence, the class number of A relative to $\mathfrak{O}$, A being considered as an A-module, coincides with the *right ideal class number of* A relative to $\mathfrak{O}$ which is defined as the number of inequivalent right $\mathfrak{O}$-ideals of A. If A is semi-simple, or more generally, if A is a Frobenius algebra, this is equal to the left ideal class number of A [3], and can be called simply the *ideal class number of* A *(relative to $\mathfrak{O}$).*

Artin [1] extended a classical result for number fields by proving that a semi-simple rational algebra has finite ideal class number relative to any maximal order. Zassenhaus [4] gave a new proof, establishing the more general result which is case (1) of the following theorem.

THEOREM. *Let $\mathfrak{o}$ be either*

(1) *the ring of integers in a finite algebraic number field, or*

(2) *the integral closure of* F[X] *in a finite extension field* k *of* F(X), *where* F *is a finite field and* X *is transcendental over* F. *Then, if $\mathfrak{O}$ is an $\mathfrak{o}$-order in a* k-*algebra* A, *every completely reducible* A-*module has finite class number relative to $\mathfrak{O}$.*

In this note we establish the result for case (2). Once it has been shown in Section 3 that a division algebra has finite ideal class number, it is possible to apply the rest of Zassenhaus's argument with only verbal changes. In Section 4 we give an alternative proof, based on a lemma from [2], of the extension from the case of irreducible A-modules to that of completely reducible A-modules.

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