A NOTE ON BOUNDED CONTINUOUS MATRIX PRODUCTS

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It is well known that the powers A^n of a complex matrix A are uniformly bounded for every positive integer n, if the eigenvalues of A have moduli less than or equal to unity and if the eigenvectors of A include a basis for the vector space. It is not generally true, however, that the products $\Pi_{i=1}^{I} A_i$ associated with an arbitrary sequence of matrices $\{A_i\}$ remain bounded even when each of the matrices has the properties mentioned. A simple counterexample is given by the matrix of the product

$$\begin{pmatrix} 1/2 & 0 \\ 10 & 1/3 \end{pmatrix}\begin{pmatrix} 1/2 & 10 \\ 0 & 1/3 \end{pmatrix},$$

which is easily shown to possess an eigenvalue greater than unity, although the eigenvalues of both factors are distinct and less than unity. In this note, sufficient conditions are established for the products $\Pi_{i=1}^{I} A(\alpha_i)$ to remain bounded when $\{\alpha_i\}$ is a sequence of values of a parameter α upon which a matrix $A(\alpha)$ depends continuously.

The problem arises in relation to the conditions for numerical stability of the solution of linear differential equations by finite difference methods. Its importance is greatest with partial differential equations, but it may be illustrated simply in the case of an equation of ordinary type. As an example, consider the equation:

$$\frac{d^2y}{dx^2} + \frac{1}{x^2}[5 + 3.66 \sin(\pi \log x)]y = 0.$$

The transformations $y = \sqrt{x}u$ and $\log x = 2\theta/\pi - 1/2$ reduce the equation to the general Mathieu form:

$$\frac{d^2u}{d\theta^2} + (1.925 - 2 \times 0.742 \cos 2\theta)u = 0,$$

which has the two independent solutions $u_1 = u_1(\theta)$ and $u_2 = u_2(\theta)$, both of which are uniformly bounded. Consequently the solutions of the original equation are

$$y = \sqrt{x}u_k\left(\frac{\pi}{4} + \frac{\pi}{2}\log x\right) \qquad (k = 1, 2),$$

and these are seen to tend to zero as x approaches the origin positively. A numerical solution is sought on the interval $0 < x \leq \sqrt{e}$ at $x_n = x_0\rho^n$ $(n = 0, 1, 2, \cdots)$ with $x_0 = \sqrt{e}$ and ρ a constant $(0 < \rho < 1)$. The following divided-difference approximation may be constructed:

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