AN HOMOLOGY ANALOGUE OF POSTNIKOV SYSTEMS

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INTRODUCTION

In [14], Postnikov presents a process for constructing semi-simplicial complexes by "adding" homotopy groups to more elementary semi-simplicial complexes. This process enables him to build a model complex equivalent to the singular complex of any given topological space. From the point of view of homotopy theory, the model is indistinguishable from the original space, but has the advantage that its structure is more conveniently displayed. The present paper studies an analogous process, in which homology groups are added to CW-complexes. This technique was used by J. C. Moore in [13], and by B. Eckmann and P. J. Hilton in their duality studies. The resulting model has the advantage that its elementary parts are complexes with relatively few cells (as compared with Postnikov complexes). On the other hand, much of the elegant algebraic structure associated with the Postnikov decomposition is lost. As an application of this homology decomposition, homotopy type classification theorems for spaces with only two nontrivial homology groups are presented. In connection with this last topic, a number of the groups [X, Y] of homotopy classes of maps are described when X and Y are spaces with at most two nontrivial homology groups.

The process for adding homology groups is dual to the Postnikov construction. B. Eckmann and P. J. Hilton have made a systematic study of this duality [6], [7], [8] and [9].

1. PRELIMINARIES

Let G be an abelian group, and let $n > 1$ be an integer. According to J. C. Moore, a topological space L has homology type (G, n) if it is simply connected, if $H_q(L) = 0$ for $q \neq 0, n$, and if $H_n(L) \approx G$. (All homology groups will be taken with integer coefficients.) It is well known that such spaces exist and that any two CW-complexes of the same type are homotopically equivalent. L(G, n) will denote the class of CW-complexes of homology type (G, n). When no confusion is likely to result, we shall also denote a member of L(G, n) by L(G, n). The following lemma can be proved by standard CW-complex arguments.

LEMMA 1.1. *If* X *is an* (n - 1)*-connected space, there exists a map* $f: L = L(H_n(X), n) \to X$ *such that* $f_*: H_n(L) \approx H_n(X)$.

Let X and Y be two spaces, and let $f: X \to Y$ be a map. The cone $\hat{X}$ over X is formed from $X \times I$ by identifying $X \times \{0\}$ to a point. Y(f)X will denote $Y \cup \hat{X}$, with f(x) and (x, 1) identified for all $x \in X$. Let $i: Y \to Y(f)X$ be the inclusion map.

LEMMA 1.2. *There is a homomorphism* α *such that the sequence*

$$\to H_n(X) \xrightarrow{f_*} H_n(Y) \xrightarrow{i_*} H_n(Y(f)X) \xrightarrow{\alpha} H_{n-1}(X) \to$$

is exact.

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