ON SINGULAR FIBERINGS BY SPHERES

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The purpose of this note is to discuss the properties of certain types of singular fibrations as introduced by Montgomery and Samelson [9]. We repeat their definition: A fiber space with singularities is a quadruple $[(X, A), (Y, B), \pi, F]$ such that

1. $\pi: (X, A) \to (Y, B)$ is a proper open onto mapping,

2. $\pi^{-1}(B) = A$ and $\pi | A$ is a homeomorphism, and

3. $\pi | X - A: X - A \to Y - B$ is a fiber mapping with fiber F.

We recall that a mapping is proper if the inverse image of every compact set is compact. This is not the only possible definition of singular fibration (see for example [10]), but it is a first step in studies related to the problems of compact transformation groups. Generally speaking, if we are given a singular fibration $[(X, A), (Y, B), \pi, F]$, the question is: what may one conclude about spaces A and Y from knowledge of X and F? As our title indicates, we are mainly concerned with the case where F is a sphere (cohomology sphere), and we obtain results closely resembling those of Smith for the stationary point set of an involution [11]. Our principal tool is of course the Gysin sequence; however, the use we shall make of it appears to be new.

This note will consider only arcwise connected, arcwise semi-locally 1-connected spaces. Although our theorems are actually valid for a fiber which is only a cohomology sphere, we shall only consider topological spheres, for convenience. We shall deal only with the Čech cohomology groups with coefficients in Z_2 (the integers mod 2), and we shall not denote the coefficient group explicitly. We shall denote the cohomology with compact supports by a subscript c [2]. In particular, we shall use $H_c^i(X - A)$ for the relative cohomology groups of the compact pair (X, A).

1. ACYCLICITY AND LOCAL CONNECTIVITY

THEOREM 1.1. *If* $[(X, A), (Y, B), \pi, S^r]$ *is a singular fibering by an* r*-sphere* $(r > 0)$ *and* X *is a compact space such that for some integer* m, $H^i(X) = 0$ *for* $i \geq m$, *then* $H^i(A) = 0$ *for* $i \geq m$, *and* $H^i(Y) \cong H^i(B)$ *for* $i \geq m - r$.

We first consider the natural diagram

$$\begin{array}{ccccccc} \to H^i(A) & \xrightarrow{\delta^*} & H_c^{i+1}(X - A) & \to & H^{i+1}(X) & \to \\ \uparrow \pi_1^* & & \uparrow \pi^* & & \uparrow & \\ \to H^i(B) & \to & H_c^{i+1}(Y - B) & \to & H^{i+1}(Y) & \to . \end{array}$$

Since $H^i(X) = 0$ $(i \geq m)$, the homomorphism δ^* is onto for $i \geq m - 1$, but π_1^* is an isomorphism; thus, π^* is onto for $i \geq m - 1$. Now we shall employ the Gysin

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