MINKOWSKI'S AND RELATED PROBLEMS FOR CONVEX SURFACES WITH BOUNDARIES

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1. INTRODUCTION

The theory of convex bodies establishes uniqueness for general closed convex hypersurfaces in E^n, when an elementary symmetric function $R_1 R_2 \cdots R_m + \cdots$ $(1 \leq m \leq n - 1)$ of the principal radii of curvature, or a suitable generalization of it in terms of set functions, is given as function of the normal. The method was initiated by Minkowski [8]. The general problem was solved independently by A. D. Alexandrov [1] and Fenchel and Jessen [5]. A convenient source in book form is provided by [3]. For surfaces of class C^2 in E^3 and given $R_1 R_2 < \infty$ Chern [4] gave a proof avoiding the Brunn-Minkowski Theory.

Also without using this theory, Hsiung [6, 7] proves the corresponding uniqueness theorems for smooth surfaces of positive curvature with a boundary in the cases: general n, m = 1, and n = 3, m = 2, following Chern's method in the latter case.

(Hsiung's historical remarks regarding the other methods cannot pass unchallenged. The decisive papers of Alexandrov, Fenchel and Jessen are not mentioned at all. Instead we find: uniqueness for closed surfaces ". . . was established by Minkowski and proved several decades later by Lewy for analytic surfaces . . . ." Lewy, as well as other cited authors, are concerned with the existence of smooth surfaces with smooth data and not with uniqueness.)

It is the purpose of this note to show that *the theory of convex bodies is applicable to surfaces with boundaries, just because it is not restricted to smooth surfaces.* In Theorem I we establish uniqueness under very wide conditions for surfaces in E^n with a given boundary and any given (generalized) $R_1 R_2 \cdots R_m + \cdots$. The completely general case would have required a modification of the theory, which will not be discussed here.

In addition we give *a uniqueness theorem and an existence theorem for convex caps* (Theorems II and III). A cap is a convex hypersurface with a closed hyperplane boundary, such that the normal projection of the surface on the plane of the boundary lies inside or on the latter. The interest in caps derives from their central role in the investigations of Pogorelov; see [3].

2. THE AREA FUNCTIONS

We use the term convex (hyper)surface as in [3] for a connected relative open subset K of a complete convex hypersurface K* in E^n. A normal to K is the unit normal to a supporting plane of K pointing into the exterior of the convex set bounded by K*. There is an ambiguity only in the trivial case, disregarded here, of a non-closed K lying in a hyperplane. The spherical image of K consists of the endpoints on the unit sphere Σ of the unit vectors beginning at the center of Σ and parallel to normals of K.

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