ON ISOMORPHISMS OF ORDERS

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1. INTRODUCTION

Let there be given a commutative ring $\mathfrak{o}$ with identity element, and an $\mathfrak{o}$-algebra $\mathfrak{O}$. As in [2], we denote by $I(\mathfrak{O})$ the ideal consisting of the elements of $\mathfrak{o}$ which annihilate the cohomology groups $H^1(\mathfrak{O}, T)$ for all two-sided $\mathfrak{O}$-modules T (cohomology being taken in the sense of $\mathfrak{o}$-algebras [1, Chapter IX]). There is a reduction theorem [1] stating that for $n > 1$, $H^n(\mathfrak{O}, T) = H^{n-1}(\mathfrak{O}, T')$ for a suitable two-sided $\mathfrak{O}$-module T'. Hence $H^n(\mathfrak{O}, T)$ is annihilated by $I(\mathfrak{O})$ for all $n > 0$.

In case $\mathfrak{o}$ is an integral domain with quotient field k, an $\mathfrak{o}$-algebra $\mathfrak{O}$ is called an *$\mathfrak{o}$-order* if it is finitely generated and torsion-free as an $\mathfrak{o}$-module. We shall call an $\mathfrak{o}$-order $\mathfrak{O}$ *separable* if its k-hull $\mathfrak{O} \otimes_{\mathfrak{o}} k$ is a separable k-algebra; a necessary and sufficient condition for this is that $I(\mathfrak{O})$ be different from 0 [2]. When $\mathfrak{O}$ is a group ring of a finite group of order N, $I(\mathfrak{O}) = N\mathfrak{o}$.

If $\mathfrak{o}$ is the valuation ring and $\mathfrak{p}$ the prime ideal of a field k with a discrete valuation, every non-zero ideal is a power of $\mathfrak{p}$, and therefore, for a separable $\mathfrak{o}$-order $\mathfrak{O}$, $I(\mathfrak{O}) = \mathfrak{p}^s$ with $s \geq 0$. We call s the *depth* of $\mathfrak{O}$.

Two $\mathfrak{o}$-orders are called *isomorphic* if there is an $\mathfrak{o}$-algebra isomorphism of the one onto the other. The purpose of this note is to prove the

THEOREM. *Let $\mathfrak{o}$ be the valuation ring and $\mathfrak{p}$ the prime ideal of a field* k *complete with respect to a discrete valuation. A separable $\mathfrak{o}$-order $\mathfrak{O}$ is isomorphic with an $\mathfrak{o}$-order $\mathfrak{Q}$ if and only if the $\mathfrak{o}/\mathfrak{o}\,\mathfrak{p}^{2s+1}$-algebras $\mathfrak{O}/\mathfrak{p}^{2s+1}\mathfrak{O}$ and $\mathfrak{Q}/\mathfrak{p}^{2s+1}\mathfrak{Q}$ are isomorphic.*

Our proof is simplified following a suggestion of the referee. The theorem reduces the problem of isomorphism of orders over complete, discrete valuation rings having finite residue class rings to a problem concerning finite algebras. Thus an immediate consequence is the

COROLLARY 1. *If $\mathfrak{o}$ as in the Theorem has finite residue class rings, there are only finitely many non-isomorphic separable $\mathfrak{o}$-orders of given finite rank and depth.*

A second corollary, concerning *genera* of orders in a separable algebra over the quotient field of a Dedekind domain $\mathfrak{o}$, is given. Here two $\mathfrak{o}$-orders are put in the same genus if their $\mathfrak{p}$-adic completions are isomorphic for each prime $\mathfrak{p}$ of $\mathfrak{o}$.

2. PROOF OF THE THEOREM

We are assuming that $\mathfrak{o}$ is the valuation ring and $\mathfrak{p}$ the prime ideal of a field k with a complete discrete valuation. Since the valuation ring $\mathfrak{o}$ is a principal ideal domain, the $\mathfrak{o}$-orders $\mathfrak{O}$ and $\mathfrak{Q}$ have free $\mathfrak{o}$-module bases. Hence an isomorphism $\mathfrak{O}/\mathfrak{p}^{2s+1}\mathfrak{O} \approx \mathfrak{Q}/\mathfrak{p}^{2s+1}\mathfrak{Q}$ is induced by an $\mathfrak{o}$-module isomorphism $\alpha\colon \mathfrak{O} \approx \mathfrak{Q}$ such that

(1) $$\alpha(xy) \equiv \alpha(x)\alpha(y) \pmod{\mathfrak{p}^{2s+1}}.$$

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