DOUBLY STOCHASTIC MEASURES

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1. INTRODUCTION

A *doubly stochastic* $n \times n$ *matrix* has nonnegative elements, and the sum of each row and each column is one. A *permutation matrix* is a doubly stochastic matrix in which the elements are zero or one. Thus the identity matrix is a permutation matrix. Many authors, including J. von Neumann [3] and G. Birkhoff [1], have shown that the set of doubly stochastic matrices is the convex hull of the permutation matrices.

We may also consider matrices with a countable infinity of rows and columns. Let $\mathcal{M}_0$ denote the space of all infinite matrices M whose row- and column-sums are absolutely convergent. We say that the net $\{M_\alpha\}$ in $\mathcal{M}_0$ converges to zero if, for each row and each column, the absolute sum converges to zero with α. With this topology, Rattray and Peck [4] have shown that the closure in $\mathcal{M}_0$ of the convex hull of the permutation matrices is the set of all doubly stochastic matrices.

2. DOUBLY STOCHASTIC MEASURES

Let X be the half-line $[0, \infty)$, and consider the quadrant $X \times X$. (We use the half-line for simplicity of notation; the results which follow hold also when X is the whole real line.) We shall denote by $\mathcal{M}$ the set of all real (not necessarily nonnegative) measures μ, defined on the measurable subsets of $X \times X$, which are such that, for each measurable set $E \subset X$ of finite Lebesgue measure $\lambda(E)$, the total variation [2, Section 39] of μ on $E \times X$ and on $X \times E$ is finite. In what follows, λ will always indicate Lebesgue measure.

If μ is a nonnegative measure on $X \times X$ such that $\mu(E \times X) = \mu(X \times E) = \lambda(E)$ for all measurable $E \subset X$, then μ will be called a *doubly stochastic measure* and $\mu \in \mathcal{M}$. A set S is a support of a nonnegative measure μ if $\mu(X \times X \sim S) = 0$. If S is a support of a doubly stochastic measure μ and is such that every section $\{y: (x, y) \in S\}$ $(x \in X)$ and $\{x: (x, y) \in S\}$ $(y \in Y)$ consists of one point only, then μ will be called a *permutation measure*. Thus the measure δ which has the diagonal $\Delta = \{(x, x): x \in X\}$ as a support and is defined by $\delta(E) = \lambda\{x: (x, x) \in E\}$ for $E \subset \Delta$ is the simplest permutation measure.

Not all doubly stochastic measures are like this. If f is a nonnegative measurable function on $X \times X$ such that $\int_X f(x, y)\, d\lambda(x) = 1$ for all y, and $\int_X f(x, y) d\lambda(y) = 1$ for all x, then f may be called a *doubly stochastic function*. For any such f, the measure μ defined by $\mu(E) = \int_E f(x, y)\, d\lambda(x, y)$ for $E \subset X \times X$ will be a doubly stochastic measure. However, δ can not be so expressed. If ψ is a nonnegative, even, measurable function on the real line such that $\int_{-\infty}^{\infty} \psi(t)\, d\lambda(t) = 1$, then $\psi(x - y)$

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