ON A THEOREM OF LEFSCHETZ

Raoul Bott

1. INTRODUCTION

This note concerns the Lefschetz hyperplane theorem in both homology and homotopy. (See Theorem I and its corollary, for the statement of the result.) We shall deduce our refinement of this oft-proved theorem (see [2], [3], [4], [6]) as an immediate consequence of what in [2] I called the main theorem of the nondegenerate Morse theory.

Morse and Lefschetz lived within a few steps of one another for over twenty years. It is therefore amusing that the idea of applying the former's theory in this connection is quite recent. I first saw this approach taken in a lecture by R. Thom in 1957, and this note is no more than a technical elaboration of his idea. My main observation is that the notion of a nondegenerate critical manifold, when properly applied, eliminates all the troubles with infinity which Thom encountered in his original account.

The proper dual of the Lefschetz theorem states that the homology of a Stein manifold vanishes above its middle dimension. In a forthcoming paper [1], Andreotti and Frankel use the Morse theory in its most elementary form to prove this dual statement. Poincaré duality then completes their proof of the classical version of the Lefschetz theorem. This approach is in a sense the simplest. But it does not yield the homotopy statement.

2. STATEMENT OF THE LEFSCHETZ THEOREM

Throughout this note, X will denote a compact, complex, analytic manifold of complex dimension n. Let E be an analytic line bundle over X. A global holomorphic section s of E will be called *nonsingular* if the following condition is satisfied:

CONDITION T. *For each* $x \in X$ *with* $s(x) = 0$, *there exist* (a) *a holomorphic section* s_* *of* E *over some neighborhood of* x *with* $s_*(x) \neq 0$, and (b) *a local analytic coordinate system* $(z_1, z_2, \cdots, z_n)$, *centered at* x, *such that near* x *the section* s *is represented by*

$$s = z_1 s_* . \tag{2.1}$$

Suppose now that s is a nonsingular section of E, and denote the set of zeros of s by S. By (2.1), this set is a closed complex analytic submanifold of X. The Lefschetz theorem compares S with X, under certain conditions on E which will be formulated next.

The fiber of E over $x \in X$ is denoted by E_x. A smooth (that is, C^∞) function $\mathfrak{h}$ which assigns to each $x \in X$ a positive definite hermitian form $\mathfrak{h}_x$ on E_x is called a hermitian structure on E. That such structures exist is easily verified. Given

Received February 25, 1959.
The author holds an Alfred P. Sloan Fellowship.