AN EXTENSION THEOREM FOR A CLASS OF DIFFERENTIAL OPERATORS

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1. INTRODUCTION. The principal theorem of this paper arises in the study of the behavior of analytic functions on the boundary of a disk, in the study of smoothing operators, and in higher-order generalizations of the Poincaré-Bendixson gradient theorem. The class of differential operators involved was first studied by Loewner [2] who showed that the curves generated by our operators (3) have the property of nonnegative circulation, that is, have nonnegative order with respect to each point. As is well known, a function of a complex variable which is analytic in a disk and continuous on the closure of the disk maps the boundary of the disk into a curve of nonnegative circulation. Later in this paper, however, we give an example of a curve of nonnegative circulation which is not such an image, even after any change of parametrization that does not change the curve's topological character. In fact, our curve is not the image of the boundary of the disk under any mapping which is light and interior on the interior of the disk, and which is thus topologically equivalent to an analytic function on the interior (Stoilow [3]). Our principal theorem shows, however, that Loewner's curves are such images; thus it proves that they form a proper subclass of the curves of nonnegative circulation. Indeed, by using a result of Jewett [1], we show that for Loewner's curves the mapping on the open disk can also be taken to be n-times differentiable.

In later work we hope to pursue this further, both in the direction of more information about the light interior function, and in the direction of integral operators.

2. Let D denote the closed unit disk in the xy-plane, and let s denote the positively oriented unit circle which bounds D.

Let $P_n(r)$ and $P_{n-1}(r)$ be a pair of polynomials with real coefficients, of degree n and n - 1, respectively:

$$(1)\qquad \begin{cases} P_n(r) = p_n^0 r^n + p_n^1 r^{n-1} + \cdots + p_n^n, \\ P_{n-1}(r) = p_{n-1}^0 r^{n-1} + p_{n-1}^1 r^{n-2} + \cdots + p_{n-1}^{n-1}, \end{cases}$$

such that

(2)

(a) $p_n^0 > 0,\ p_{n-1}^0 > 0;$

(b) the roots of P_n and P_{n-1} are real and simple;

(c) the roots of P_{n-1} separate the roots of P_n; that is, if r_n^i and r_{n-1}^i are the roots of P_n and P_{n-1}, respectively, then $r_n^1 < r_{n-1}^1 < r_n^2 < \cdots < r_{n-1}^{n-1} < r_n^n$.

Let f(t) be a real-valued function in C^{n+1} defined over s, where t is the real-angle parameter $(0 \leq t \leq 2\pi)$. Consider the pair of differential operators obtained from polynomials in (1) as applied to the function f(t):

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