ON SUBFACTORS OF FACTORS OF TYPE II_1

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1. INTRODUCTION

In the series of papers entitled *On Rings of Operators*, Murray and von Neumann study certain classes of operator algebras on a Hilbert space $\mathscr{H}$. Among the more remarkable types of algebras are the factors of type II_1 [3, p. 172] which, although they have infinitely many orthogonal nonzero projections (self-adjoint idempotents), have a unique linear functional tr such that

(1) $\mathrm{tr}(AB) = \mathrm{tr}(BA)$,

(2) $\mathrm{tr}(A^*A) \geq 0$, and $\mathrm{tr}(A^*A) = 0$ only if $A = 0$,

(3) $\mathrm{tr}(I) = 1$, where I is the identity operator.

An $f \in \mathscr{H}$ such that $\mathrm{tr}(A) = \alpha(Af, f)$ for $\alpha > 0$ will be called a trace vector. Although Murray and von Neumann assume $\mathscr{H}$ to be separable, subsequent work has shown this assumption to be unnecessary, and most proofs in [3], [4], [5] do not assume separability of $\mathscr{H}$. Therefore, the definitions and notation of [3] will be used, except that factors will be designated by script letters. All isomorphisms mentioned will preserve the adjoint operation.

In [5, Section 5.3] it is shown that any (countable) group G whose non-identity conjugate classes contain infinitely many elements will lead to a factor of type II_1 on a (separable) Hilbert space. In this paper, we study relationships between a II_1-factor and a II_1-subfactor which are reminiscent of group and subgroup relationships. The work was motivated by the factors generated in the manner of [5] by the group of all finite permutations of the integers and the subgroup of all even permutations.

First we select the factors to be studied. In [4, Theorem II], it is shown that a II_1-factor $\mathscr{M}$ with a vector cyclic under $\mathscr{M}$ and $\mathscr{M}'$ (we use the superscript $'$ to denote the commutant [3, p. 117]) possesses a trace vector $f \in \mathscr{H}$ with $\|f\| = 1$. Associated with f, which will now be fixed, is an anti-isomorphism $A \to A'$ for $A \in \mathscr{M}$, $A' \in \mathscr{M}'$ defined by $Af = A'f$. Hence, $\mathrm{tr}_{\mathscr{M}'}(A') = (A'f, f)$. The details of the anti-isomorphism are in [4, Chapter IV]. Let $\mathscr{A} \subset \mathscr{M}'$ be a II_1-subfactor such that $\mathscr{A}'$ is finite. Let

$$c_1 = \dim_{\mathscr{A}'}([\mathscr{A}f]),$$

where $[\mathscr{S}f]$ = closure of $\{Sf: S \in \mathscr{S}\}$. Let $\mathscr{N}' = \{A' \in \mathscr{M}': A'f = Af \text{ for } A \in \mathscr{A} \subset \mathscr{M}\}$. Then $\mathscr{N}'$ is anti-isomorphic to $\mathscr{A}$, and is weakly closed by [5, p. 728]. Let $c_2 = \dim_{\mathscr{N}}([\mathscr{N}'f])$. We shall show that $c_1 = c$. Since $\mathscr{A} \subset \mathscr{M}$, any trace vector for $\mathscr{M}$ will be a trace vector for $\mathscr{A}$, but $\mathscr{A}$ will have trace vectors which are not trace vectors for $\mathscr{M}$. We shall study trace vectors g for $\mathscr{A}$ which lie in the "smallest" subspaces $\eta\,\mathscr{M}'$ in which such trace vectors can lie, that is, in subspaces of dimension c_1 by [3, Lemma 9.3.3]. Theorem 1 shows that $g = \alpha Vf$, where $V \in \mathscr{M}$ is a partial isometry with $\dim(V^*V) = c_1$. If $c_1 = 1/n$ for integral n and there are "enough" different V's giving trace vectors for $\mathscr{A}$, then there is a coset-like decomposition of $\mathscr{H} = [\mathscr{A}f_1] \oplus \cdots \oplus [\mathscr{A}f_n]$, where f_k is a trace vector for $\mathscr{M}$. If

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