COMPARISON OF SINGULAR AND ČECH HOMOLOGY IN LOCALLY CONNECTED SPACES

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The main result of this paper is Theorem 1 (see Section 2). It shows that a natural homomorphism ν_* (ν^*) of the singular homology (Čech cohomology) theory into the Čech homology (singular cohomology) theory becomes an isomorphism in dimensions $0 \leq q \leq p + 1$, provided that we restrict ourselves to the category of paracompact Hausdorff spaces which are lc_s^p and semi-$(p + 1)$-lc_s (for these notations see Section 2). The proof is carried out in Sections 1 to 5.

In the case of triangulable spaces, the equivalence of singular and Čech theory is a well-known fact (a consequence of the uniqueness theorem of S. Eilenberg and N. Steenrod [4]). The same fact has been established more recently for ANR-s (see J. Dugundji [3], Y. Kodama [9] and S. Mardešić [11]). For metrizable compacta which are homotopy locally connected, the equivalence has been established by S. Lefschetz ([10], p. 107). A proposition, closely related to the part of Theorem 1 which is concerned with cohomology, can be derived from Cartan's uniqueness theorem for cohomology with coefficients in sheaves [1]; this approach is not applicable to homology. Finally, Theorem 1 generalizes a result obtained by H. B. Griffiths in [6]. (In an unpublished paper, Griffiths has recently developed a general theory of "locally trivial homology," and Theorem 1 is there derived in the framework of that theory.)

Section 7 contains a proof of an analogue of Theorem 1, for Čech homology with compact carriers. In Section 8, we show that, for locally paracompact spaces, $lc_s^p \Rightarrow lc_c^p$ (for notation see Section 2). In Section 9, an application of Theorem 1 gives a criterion for unicoherence of locally arcwise connected semi-1-lc_s paracompact Hausdorff spaces in terms of the first singular homology group.

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1. NATURAL HOMOMORPHISMS ν_* AND ν^*

Let (X, A) be a pair of topological spaces, $A \subset X$ (A need not be closed). $H_q(X, A; G; S)$ and $H^q(X, A; G; S)$ will denote the q-th singular homology group and the q-th singular cohomology group, both taken with coefficients in a (discrete) group G. The corresponding Čech groups will be distinguished by a letter C replacing S. For purposes of this paper, we adopt a definition of Čech homology and cohomology which naturally generalizes the classical notions of the Vietoris theory, and which was introduced by E. Spanier in [13] and by C. H. Dowker in [2]. The definition gives groups which are naturally isomorphic to usual Čech groups for arbitrary pairs (X, A) (for a proof see [2]). Here is the Spanier-Dowker definition.

Let $\alpha = (\alpha_1, \alpha_2)$ be an open covering of (X, A). This means that α_1 is an open covering of X, that $\alpha_2 \subset \alpha_1$ and that the union of all the members of α_2 is a subset of X which contains A. Let $K_{\alpha_1}(X)$ ($K_{\alpha_2}(A)$) be the simplicial complex whose vertices are all the points of X (of A); a finite set of vertices forms a simplex of

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