SOME OPAQUE SUBSETS OF A SQUARE

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We deal in this note with a fixed Euclidean plane. Let

$$Q = \{(x, y): 0 \leq x \leq 1,\ 0 \leq y \leq 1\}.$$

We say that a set S is *opaque*, if (i) S is a subset of Q, and (ii) every (straight) line that contains a point of Q also contains a point of S. If T is a subset of Q, the *distance set*, Δ_T, of T is defined to be the set of all real numbers d with the property that there exist points t, t' in T such that the distance between t and t' is d. Let J be the closed interval $[0, \sqrt{2}]$, and let $J^* = J - \{0, 1, \sqrt{2}\}$. If $T \subset Q$, then clearly $\Delta_T \subset J$, and if T is opaque, then $\Delta_T \supset \{0, 1, \sqrt{2}\}$ because T contains the vertices of Q.

A recent article by Sen Gupta and Basu Mazumdar [8] is devoted to showing that there exists a subset E of Q of first category and measure zero such that (a) every line that contains a point of Q and is not parallel to a side of Q also contains a point of E, and (b) $\Delta_E = J$, and if $0 < d < \sqrt{2}$ then there are infinitely many pairs of points in E such that the distance between the points of each pair is d. We remark that there is a very much simpler example of such a set E: the union of the two diagonals of Q not only satisfies (a) and (b), but is actually opaque, and is obviously a *nowhere dense* perfect subset of Q of measure zero.

There are perfect opaque sets that are even *punctiform* (that is, they contain no continuum having more than one point); in fact, Mazurkiewicz showed [6] that every polygon (to which we reckon interior points as well as frontier points) has a perfect punctiform subset that intersects every line that meets the polygon. We shall describe a perfect, punctiform, opaque set whose construction is akin to that of Mazurkiewicz but which is somewhat easier to see and to remember.

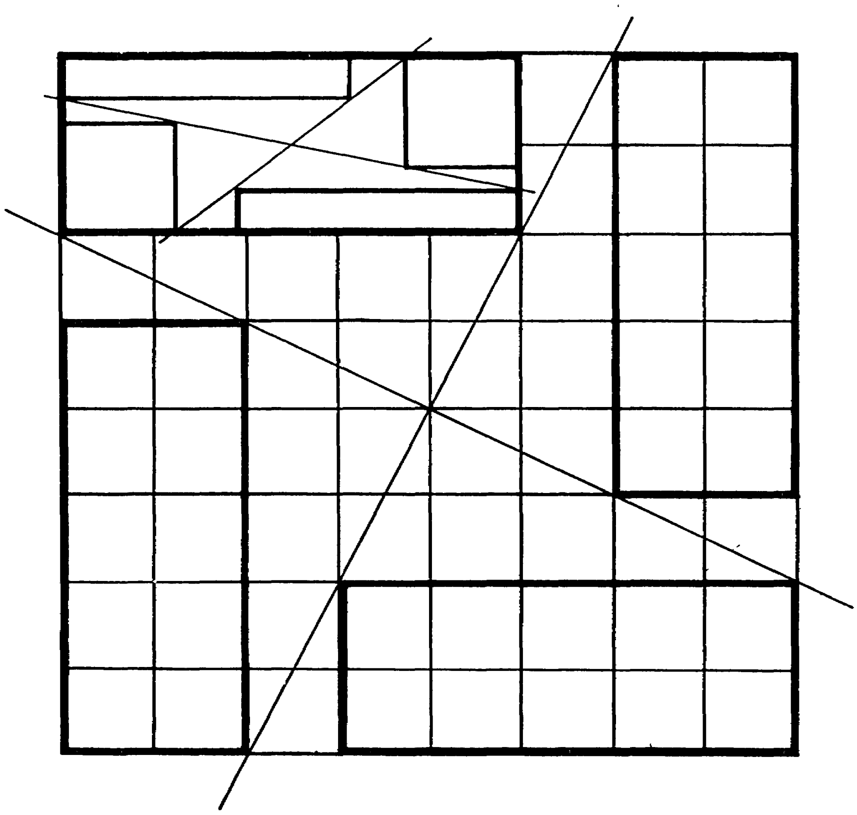

Figure 1

We begin (see Fig. 1) by dividing each side of Q into eight equal segments, thereby inducing a division

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