AN INTEGRAL INVOLVING THE ASSOCIATED LEGENDRE FUNCTIONS OF THE FIRST KIND

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1. INTRODUCTION

The formula

$$
(1)\qquad \begin{aligned}
&\int_0^\infty (\sinh u)^{m+1}\, P_n^{-m}(\cosh u)\, (\cosh u)^{-\ell}\, E(p;\alpha_r\colon q;\rho_s\colon z\cosh^2 u)\, du \\
&= 2^{-m-1}\, E\left[\begin{matrix} \alpha_1, \cdots, \alpha_\rho, \frac{\ell - m + n}{2}, \frac{\ell - m - n - 1}{2}; \\ \rho_1, \cdots, \rho_q, \frac{\ell}{2}, \frac{\ell + 1}{2}; \end{matrix} \; z\right]
\end{aligned}
$$

will be proved under the conditions

$$\Re(\ell - m + n) > 0, \qquad \Re(\ell - m - n) > 1, \qquad \Re(m) > -1;$$

the functions P_n^{-m} and E under the integral sign are defined by the relations

$$
(2)\qquad P_n^{-m}(t) = \frac{2^{-m}(t^2 - 1)^{m/2}\, t^{n-m}}{\Gamma(m+1)}\, {}_2F_1\left[\begin{matrix} \frac{m-n+1}{2}, \frac{m-n}{2}; \\ m+1; \end{matrix}\; \frac{t^2-1}{t^2}\right],
$$

$$
(3)\qquad E(p;\alpha_r\colon q;\rho_s\colon z) = \frac{\Gamma(\alpha_1)\cdots\Gamma(\alpha_p)}{\Gamma(\rho_1)\cdots\Gamma(\rho_q)}\, {}_pF_q\left[\begin{matrix} \alpha_1, \cdots, \alpha_p; \\ \rho_1, \cdots, \rho_q; \end{matrix}\; \frac{-1}{z}\right] \qquad (p \leq q);
$$

for a discussion of the E-functions, see [3, p. 352].

From formula (1), integration formulas involving many special functions (for example, those of Bessel, Whittaker, Struve) can be deduced. The formulas thus obtained provide information on the asymptotic behavior of the integrals for large values of $|z|$; for the asymptotic expansion of the E-functions is given by MacRobert [3, p. 358], and the other functions that occur can be expressed in terms of ordinary generalized hypergeometric functions whose asymptotic expansion has been investigated by several writers (Barnes [1], Wright [5], Meijer [4]).

Received February 21, 1957.