TRANSIENT FLOWS IN NETWORKS

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1. INTRODUCTION

Ford and Fulkerson [1] have introduced the notion of dynamic flows in networks. A *dynamic network* consists of a graph Γ to each edge e of which corresponds a nonnegative integer $\gamma(e)$, called the *capacity* of the edge, and a second nonnegative integer $\tau(e)$, called the *transit time* of the edge. In terms of transportation networks, the capacity γ is to be thought of as giving an upper bound to the amount that can be shipped along an edge e, while the transit time τ specifies how long it takes a shipment to traverse this edge. In this framework, Ford and Fulkerson have considered the following problem: For a dynamic network Γ with two distinguished terminals s and s' (called the *source* and the *sink*, respectively), to determine the maximum amount μ_k that can be shipped from s to s' in k time periods. In the work referred to, the authors describe an ingenious algorithm for obtaining μ_k for each integer k. More precisely, they show, for each integer k, how to obtain a *flow* ϕ_k (to be thought of as a shipping schedule) that achieves the desired shipment μ_k from s to s'.

Concerning the solution of Ford and Fulkerson, the following observation may be made. In order to achieve the maximum numbers $\mu_1, \mu_2, \cdots, \mu_k$, the authors construct a sequence of flows $\phi_1, \phi_2, \cdots, \phi_k$. It would be computationally advantageous if it turned out that ϕ_2 is a "continuation" of ϕ_1 and, in general, ϕ_{i+1} a continuation of ϕ_i. Put another way, one might hope that the flow ϕ_k has the property that for each time $i < k$ the amount already shipped into s' is the maximum μ_i. In this case the single flow ϕ_k would provide a solution to the maximum problem, not only for k time periods, but also for any smaller number of periods. However, the flows obtained by the authors do not have this desirable property; indeed, it is not clear from their work that such *universal maximal flows* exist. It is our purpose here to show that they exist, not only for the case treated by Ford and Fulkerson, but also for the considerably more general case in which the capacities γ and transit times τ may vary with time.

2. A LEMMA ON STATIC NETWORK FLOWS

The result needed for proving the main theorem of this paper (see Section 3) is the Feasibility Theorem obtained by the author in [2]. We shall here record the definitions needed for a statement of that result. For motivation and interpretation of these definitions, the reader is referred to [2].

A *network with a source* is a triple $[X, s; \gamma]$, where X is a finite set of elements x, y, $\cdots$, called *nodes;* s is a distinguished node of X, called the *source;* and γ, the *capacity* of the network, is a function on pairs (x, y) of nodes, such that $\gamma(x, y)$ is a nonnegative integer or plus infinity.

A *flow* ϕ on X is a function from ordered pairs (x, y) to the integers satisfying the conditions

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