THE FOURIER SERIES OF FUNCTIONS OF BOUNDED pTH POWER VARIATION

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1. A function f(x) is said to be of bounded pth power variation in [a, b], or to be of Wiener class $W_p(a, b)$, provided the upper bound $V_p(f; a, b)$ of the expression

$$\left(\sum_r |f(x_r) - f(x_{r-1})|^p\right)^{1/p},$$

taken with respect to all subdivisions $a = x_0 \leq x_1 \leq \cdots \leq x_n = b$, is finite. We state some relevant known facts as lemmas.

LEMMA 1 (see [3, p. 259]). *The quantity* $V_p(f; a, b)$ *is a decreasing function of* p, *and for all* $q \geq p$,

$$V_p(f; a, b) \leq V_p^{q/p}(f; a, b)\,[\mathrm{Osc}\,(f; a, b)]^{(q-p)/q},$$

where Osc (f; a, b) *denotes the oscillation of* f(x) *in* (a, b).

In connection with Stieltjes integration and functions of class W_p, L. C. Young has established the following inequality of Hölder type [3, p. 266].

LEMMA 2. *Let* $f \in W_p(a, b)$ *and* $g \in W_q(a, b)$, *where* $p > 1$, $q > 1$, *and* $s = 1/p + 1/q > 1$. *If* f *and* g *have no common discontinuity in* (a, b), *then the Riemann-Stieltjes integral* $\int_a^b f(x)\,dg(x)$ *exists, and for each* η *in* [a, b],

$$\left|\int_a^b [f(x) - f(\eta)]\,dg(x)\right| \leq [1 + \zeta(s)]\,V_p(f; a, b)\,V_p(g; a, b),$$

where $\zeta(s) = \Sigma_{n=1}^{\infty} n^{-s}$.

LEMMA 3. *If* $p > 1$, $f \in W_p(a, b)$, $g \in W_p(a, b)$, *and* $h(x) = f(x)g(x)$, *then* $h(x) \in W_p(a, b)$.

Proof. By hypothesis, $|f(x)|$ and $|g(x)|$ are bounded by some constant K. On writing

$$f(x_r)g(x_r) - f(x_{r-1})g(x_{r-1}) = [f(x_r) - f(x_{r-1})]g(x_r) + f(x_{r-1})[g(x_r) - g(x_{r-1})]$$

and applying Minkowski's inequality, we obtain the relation

$$\left(\sum_r |f(x_r)g(x_r) - f(x_{r-1})g(x_{r-1})|^p\right)^{1/p}$$
$$\leq K\left(\sum_r |f(x_r) - f(x_{r-1})|^p\right)^{1/p} + K\left(\sum_r |g(x_r) - g(x_{r-1})|^p\right)^{1/p}.$$

The lemma now follows on taking the upper bound on both sides.

Received July 10, 1957.