A THEOREM ON NONLOXODROMIC MÖBIUS TRANSFORMATIONS

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If a Möbius transformation has one fixed circle, it has a coaxal family of fixed circles. The theorem I shall prove relates the mapping on any fixed circle to the axis of this family, using the real projective structure of the plane.

THEOREM 1: *If* g *is a fixed circle of a Möbius transformation* T *and the distinct points* A, B, C *have images* A', B', C' *on* g, *then the axis of the system of fixed circles of* T *is the Pascal line of the hexagon* AB'CA'BC'.

The proof makes use of the theory of projectivities on a conic (see [1], Chap. VII). This theory is based upon the fact that if A, B, C, D are points on a conic and P is a variable point on it, then the cross-ratio (PA PB, PC PD) is constant. Using this four-point invariant, we can define projective correspondences on the conic, and it is clear that any projectivity on a line in the plane of the conic can be mapped onto the conic from any point of the latter with preservation of its character: that is, fixed points map onto fixed points, and the invariant cross-ratio of a point and its image with respect to the fixed points is preserved. A pretty result of this theory is the following.

THEOREM 2. *If* J *is a projectivity on a conic and* A', B', C' *are the images of distinct points* A, B, C *under* J, *then the Pascal line of the hexagon* AB'CA'BC' *meets the conic in the fixed points of* J.

For a real conic this theorem classifies the projectivity: J is hyperbolic, parabolic or elliptic according as the line cuts the conic in 2, 1 or 0 points. The classification of nonloxodromic Möbius transformations by the relation of the fixed axis to the fixed circles is analogous (see [2], Chap. 1).

Let us consider Theorem 1 in the hyperbolic case, the distinct fixed points being M and N. The family G of fixed circles is the family on M and N; the axis u is the line MN. The cross-ratio (Z T(Z), M N) = λ is real. To see that the transformation on g is a projectivity in the sense of the above discussion, we note that the mapping on u is a real projectivity, and that if P is a point of g other than M or N, then (PZ PT(Z), PM PN) = λ for Z either on u or on g. Thus the pencil at P relates the transformations as required; the circle is a conic on which Theorem 2 holds, and Theorem 1 follows for this case.

If T is parabolic, u is the tangent to g at the single fixed point M. We may argue as before if, instead of λ as defined above, we use the fact that in this case (Z T^2(Z), T(Z) M) = -1, this being necessary and sufficient for a projectivity (real or complex) to have no other fixed point.

In the elliptic case we proceed differently. Here $\lambda = e^{i\theta}$, and u is the perpendicular bisector of the segment MN. Let W be the coaxal system on M and N; all circles of W are then orthogonal to g, and M and N are the "limiting points" of G. Let us say M lies within g; the angle at M between the arc MA and its image arc MA' is θ.

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