A NOTE ON THE SYSTEM GENERATED BY A SET OF ENDOMORPHISMS OF A GROUP

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The study of a set $\mathfrak{E}$ of endomorphisms of a group G has been limited generally to the case of an abelian group G, although the set $\mathfrak{E}_1$ of all normal endomorphisms of a nonabelian group G has been studied by Fitting and others (see [3], [4]). In the abelian case, a ring R can be formed from $\mathfrak{E}$ and studied instead of $\mathfrak{E}$. In similar fashion a type of near-ring (a distributively-generated near-ring) N can be formed from $\mathfrak{E}$ in the general case, and it is the purpose of this note to develop a structure theory for these near-rings which generalizes the Artin-Wedderburn theory for rings. The development is kept brief, both because of the analogy and because of the existence of some information on general near-rings (see [1], [2], [5]). Certain distinctions between the rings and the non-ring near-rings are discussed in the final section.

1. PRELIMINARY REMARKS

Let $\mathfrak{E}$ be a set of endomorphisms of an additively-written group G which satisfies the DCC on $\mathfrak{E}$-subgroups. Addition and multiplication of endomorphisms E and F of G are defined by the equations

$$g(E + F) = gE + gF \quad \text{and} \quad g(EF) = (gE)F \quad (g \in G).$$

Extend the set $\mathfrak{E}$ to the semigroup $\mathfrak{E}'$ of all products of finitely many elements of $\mathfrak{E}$. Then the subset $R(\mathfrak{E})$ of the set of all mappings of G into itself consisting of all finite linear combinations $\Sigma r_i E_i$ of elements E_i of $\mathfrak{E}'$ with rational integral coefficients r_i will be called the *system generated by* $\mathfrak{E}$.

Now a *near-ring* N is a set of elements with two binary operations, written as addition and multiplication, such that

i) N is a group relative to addition;

ii) N is a semigroup relative to multiplication;

iii) $a(b + c) = ab + ac$ for all $a, b, c \in N$.

An additive subgroup M of N is called a *right module* provided $MN \subseteq M$. A near-ring N which

i) contains a multiplicative semigroup D of right distributive elements d ($(b + c)d = bd + cd$ for all $b, c \in N$) such that each element of N can be written as a finite linear combination $\Sigma r_i d_i$ of d_i of D with rational integral coefficients r_i,

ii) satisfies the DCC for right modules

is called a *distributively-generated* near-ring (DGN-ring). Obviously $R(\mathfrak{E})$ is a DGN-ring. If the additive group of a DGN-ring is denoted by G and D by $\mathfrak{E}$, then clearly the system $R(\mathfrak{E})$ is a homomorphic image of N (right regular representation) and is isomorphic with N if, for instance, N has a multiplicative identity.

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