A CHARACTERIZATION OF GENERALIZED MANIFOLDS

P. E. Conner and E. E. Floyd

1. INTRODUCTION

We are concerned here with generalized manifolds; these are spaces which have certain of the local homological (equivalently: cohomological) properties of manifolds. Such spaces have been of interest in topology since the fundamental work of Wilder [7], as well as that of Lefschetz, Čech, Smith, Begle, and others. Recently, there has been renewed interest in such spaces; see papers of Floyd [5], Yang [9], and Borel [2]. The principal motivation for the present paper is the study of fixed point sets of toral groups of transformations. In a later paper, we shall apply our results.

We consider locally orientable generalized manifolds in the sense of Wilder [7]. In our terminology, a finite-dimensional space X is a *locally orientable generalized* n-*manifold* if it fulfills two requirements: (a) X must have local cohomology modules $H_x^*(X, K)$ over the coefficient ring K at each point x, and their union must have the structure of a locally constant sheaf; (b) $H_x^i(X, K) = 0$ for $i \neq n$ and $H_x^n(X, K) \approx K$.

In Section 4, we show that X has locally constant local cohomology modules if and only if X has property Q of P. A. Smith [6]; that is, requirement (a) above is equivalent to property Q. Using recent work of Borel [2] and an argument on spectral sequences, we show in Section 6 that condition (a) implies condition (b) when the coefficients are in a field. That is, if X is connected, finite-dimensional and has locally constant local cohomology modules, then X is a locally orientable generalized n-manifold, for some n. Otherwise stated, the properties P and Q of P. A. Smith [6] are equivalent, at least when coefficients are in a field. Yang [9] has already proved, for general K, that P implies Q. In Section 7, we prove slightly weaker theorems for the case where the coefficients are in the ring Z of integers. The problem of determining whether the above results hold in full generality for this case is left unsolved.

Neighborhoods of points will always be open sets in the space. Unless otherwise specified, the coefficient ring K will be assumed to be an arbitrary commutative ring consisting of more than one element. The kernel of a homomorphism $f: F \to G$ is denoted by Ker f, and its image by Im f.

2. INVERSE FAMILIES

Suppose that X is a topological space. Let there be given for each open U in X a K-module F_u, and whenever $U \supset V$, let there be given a homomorphism

$$f_{uv}: F_v \to F_u$$

such that (1) f_{uu} is the identity and (2) $f_{uv} f_{vw} = f_{uw}$ whenever $U \supset V \supset W$. Then we say that $[F_u, f_{uv}]$ is *an inverse family on the space* X. The following concepts

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