BINARY OPERATIONS ON SETS OF MAPPING CLASSES

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1. INTRODUCTION

1.1. SUMMARY OF RESULTS

Suppose that X and Y are topological spaces, with points $x_0 \in X$ and $y_0 \in Y$. The symbol $[X, x_0; Y, y_0]$ will be used to denote the set of homotopy classes of maps (continuous functions) from (X, x_0) to (Y, y_0). The chief purpose of this paper is to investigate two ways in which $[X, x_0; Y, y_0]$ may be equipped with a binary operation. The first way (homotopy theory) imitates the Hurewicz homotopy theory; (X, x_0) is held fixed, and the binary operations are so chosen that the class of the constant map is an identity element, and every map from a pair (Y, y_0) to a pair (Y', y_0') induces a homomorphism. It is shown that the binary operations exist if and only if (X, x_0) has (modified) Lusternik-Schnirelmann category at most two. The second way (cohomotopy theory) imitates the Borsuk-Spanier cohomotopy theory, but without dimensional restrictions. In this case, the existence of binary operations is related to the fact that Y is an H-space. For each of these theories, exact sequences of a natural sort are developed, together with certain lesser results.

The last section contains applications of the earlier parts of the paper. A result of Spanier and J. H. C. Whitehead (roughly, that a fibre contractible in its fibre space is an H-space) is presented in a somewhat strengthened form. Several results in the direction of describing $[X, x_0; Y, y_0]$ when X and Y are "simple" spaces are obtained. Finally, the problem of determining the structure maps on an H-space is solved in case the space has only two nontrivial homotopy groups.

1.2. NOTATION AND CONVENTIONS

All of the spaces considered in this paper are Hausdorff spaces. The unit interval $[0, 1]$ is denoted by I; the n-cube $I \times \cdots \times I$ (n factors), by I^n. The boundary of I^n is designated by $\dot{I}^n$.

If X and Y are spaces, or topological pairs or triples, then $[X; Y]$ is the set of homotopy classes of maps from X into Y. If $f: X \to Y$ and $g: Y \to Z$ are maps, then $g \circ f: X \to Z$ is the map given by $g \circ f(x) = g(f(x))$ for $x \in X$. The symbol $\{f\}$ denotes the homotopy class of the map f. A map $f: X \to Y$ induces functions

$$f_{\#}: [Z; X] \to [Z; Y] \quad \text{and} \quad f^{\#}: [Y; Z] \to [X; Z];$$

these functions are defined by $f_{\#}\{g\} = \{f \circ g\}$ when $\{g\} \in [Z; X]$, and by $f^{\#}\{g\} = \{g \circ f\}$ when $\{g\} \in [Y; Z]$.

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