A REAL CONTINUOUS FUNCTION ON A SPACE ADMITTING TWO PERIODIC HOMEOMORPHISMS

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This note is concerned with a map invariant under the even powers of two periodic fixed-point-free homeomorphisms of even periods $2m_1$ and $2m_2$. A special case with $m_1 = m_2 = 1$ is known [3], also a case with $m_1 = m_2 = 2$ [1]. The latter is of interest as the basis of the proof in [1] that the intersection of the boundary of a strictly convex set in Euclidean 3-space and some sphere contains the endpoints of three orthogonal diameters. (Presumably, the boundary of a convex body in E^n contains the endpoints of n mutually orthogonal diameters [of equal lengths] through some point; but the writer has not yet succeeded in proving this.)

A covering space of Y is a pair (P, p), where P is a connected, locally connected Hausdorff space, and where p is a (projection) map of P onto Y such that each $y \in Y$ admits a connected *evenly* covered open neighborhood [2, p. 41]. Such open neighborhoods are called *preferred sets*. By a measure μ we shall mean a monotone set function to the nonnegative reals which is defined for all open and closed subsets, is at least finitely additive on disjunct open or closed sets, and has the property $\infty > \mu(P) \geq \mu(O) > 0$, for each open set O. Write C(2m) for the cyclic group of order 2m, $\{R^n | n = 0, \cdots, 2m - 1\}$. The elements $R^{2\ell}$ ($\ell = 0, 1, 2, \cdots$) are the *even* elements, and the elements $R^{2\ell+1}$ are the *odd* elements of the group.

We shall say that y_1, y_2 in Y are *chained* if there is a finite ordered collection of preferred sets $\{O_i | i = 1, \cdots, n\}$ such that $O_i \cap O_{i+1}$ is not empty and $y_1 \in O_1$, $y_2 \in O_n$. If $\{O_i\} \subset U$, then y_1 and y_2 are *chained in* U.

THEOREM. *Let* P *be a unicoherent locally connected compactum with points* $\{r\}$. *Suppose* P *admits a measure* μ, *and suppose* R_1 *and* R_2 *are two measure-preserving homeomorphisms of minimum periods* $2m_1$ *and* $2m_2$, *respectively, with* R_i^j $(j \leq m_i)$ *fixed-point-free. Let* f *be a continuous real-valued function on* P, *subject to the condition*

$$(\alpha) \qquad f(rR_i^{2\ell_i}) = f(r) \qquad (\ell_i = 1, \cdots, m_i - 1;\ i = 1, 2).$$

Then, for some $\bar{r} \in P$, $f(\bar{r}) = f(\bar{r}R_1) = f(\bar{r}R_2)$.

We follow, in general, the ideas of our proof of the special case where P is projective 3-space [1, Theorem 3]. For each $r \in P$, the transforms $r, rR_i^2, \cdots, rR_i^{2m_i-2}$ are distinct, since, after reduction mod $2m_i$, $0 \leq \min(s_i, m_i - s_i) \leq m$. Accordingly, let Y_i be the identification space with points

$$y = y(r) = (r, rR_i^2, \cdots, rR_i^{2m_i-2}).$$

Let T_i project P on Y_i by $T_i r = y = y(r)$ $(= y(rR_i^{2\ell_i})$. Write R, Y, T for the corresponding entities either with the subscript 1 or the subscript 2. The topology of Y is defined by taking V as open in Y if and only if $T^{-1}V$ is open in P. By compactness, $\inf d(r, \bigcup_{j=1}^{j=2m-1} rR^j) > 0$, where d(,) is the metric. Hence (P, T),

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