HYPERCOMPLETE LINEAR TOPOLOGICAL SPACES

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The classical closed-graph theorem of Banach states that a linear transformation T of a complete metric linear space E into a second space F of the same type is continuous whenever it has a closed graph. More generally, T is continuous provided the inverse of each nonvoid open set in F is somewhere dense, T has a closed graph, E is metrizable, and F is a complete metric space. This theorem does not hold for complete linear topological spaces E and F, as the following example shows. If E is any infinite-dimensional Banach space and F is the same space with the strongest (largest) locally convex Hausdorff topology, then the identity map T has the property that the inverse of each nonvoid open set is somewhere dense (by way of a simple category argument), and T surely has a closed graph since T^{-1} is continuous. Yet T is not continuous, since F is not metrizable. Moreover, it is easily seen that in this case F is complete, *tonnelé*, reflexive and bornological because F is isomorphic to a direct sum of copies of the scalar field. Thus the validity of the closed-graph theorem requires a stronger hypothesis than any of the usual linear topological space specifications.

The purpose of the investigation reported here is to describe the class of locally convex Hausdorff spaces F for which a closed-graph theorem holds for all possible spaces E (a more precise statement is given later). It will be shown that such spaces F are precisely those which satisfy a weakened form of the requirement that the class $\mathscr{C}$ of all convex circled subsets of F be complete relative to the Hausdorff uniformity. We shall call F hypercomplete if $\mathscr{C}$ is complete; a complete metric space is automatically hypercomplete. It will be shown that F is hypercomplete if and only if each convex circled subset A of the adjoint F* is weak*-closed whenever its intersection with each equicontinuous set B is closed in B; that is, hypercompleteness is equivalent to the well-known theorem of Krein and Šmulian.

Thus both the closed-graph theorem and the Krein-Šmulian theorem hold for hypercomplete spaces. These two propositions are perhaps the most striking consequences of a nonmetric completeness requirement that have been attained. Indeed, the general notion of completeness has played a very disappointing role in linear topological space theory in contrast to its basic importance in normed space theory. It now appears that, whereas the class of *tonnelé* spaces is the natural extension of the second category class, hypercomplete spaces are in similar position with respect to complete metric spaces.

Previous work in this direction includes the following. Ptak [5] showed that the open-mapping theorem for F (see Theorem 2) is equivalent to the property (W) that a subset of F* be weak*-closed if its intersection with each equicontinuous subspace B of F* is weak*-closed in B. Collins [2] studied property (W) further, and established some permanence properties. Recently A. P. and W. Robertson [6] proved the closed-graph theorem for E *tonnelé* and F with property (W).

There are several open questions on hypercompleteness. The first of these concerns the permanence properties on which much of the usefulness of the notion depends. It will be shown in what follows that closed subspaces and quotients of

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