HERMITIAN MANIFOLDS WITH ZERO CURVATURE

William M. Boothby

1. INTRODUCTION

In this note we consider the problem of determining those complex-analytic manifolds with a Hermitian metric whose curvature vanishes everywhere. It is easy to see that the identical vanishing of the curvature implies that there exists in a neighborhood of each point a field of n independent (in fact, orthonormal) parallel analytic vectors, where n is the dimension of the manifold. If the manifold is simply connected, such a field may then be defined over the entire manifold, and the manifold is therefore parallelisable (a complex-analytic manifold of complex dimension n is said to be parallelisable if there exist n analytic vector fields defined over it which are independent at each point). On the other hand, if a complex-analytic manifold is parallelisable, then it has a Hermitian metric with curvature zero. Hence, for a complex-analytic manifold, the existence of such a metric is a somewhat weaker property than parallelisability. H. C. Wang [6] has shown that a compact, complex-analytic, parallelisable manifold has a complex Lie group as its universal covering space. Here this is generalized to the corresponding theorem for the case of vanishing curvature.

We use the notation of [4], except that we denote the conjugate of a complex number by a bar, and that the * on indices is replaced by a bar. Thus Greek indices range from 1 to 2n, unbarred Latin indices from 1 to n, and barred Latin indices from n + 1 to 2n. In local coordinates $z^1, \cdots, z^n$, and relative to the natural (affine) frames, the metric tensor is denoted by $g_{i\bar{j}}\, dz^i d\bar{z}^j$, and the components of the connection $C_{\beta\gamma}^{\alpha}$ are given by

$$C_{jk}^{i} = g^{i\bar{l}} \frac{\partial g_{j\bar{l}}}{\partial z^k}, \qquad C_{\bar{j}\bar{k}}^{\bar{i}} = \bar{C}_{jk}^{i},$$

all other components being zero. The torsion tensor is simply the skew-symmetric part of the connection, that is, $A_{\beta\gamma}^{\alpha} = C_{\beta\gamma}^{\alpha} - C_{\gamma\beta}^{\alpha}$. Its vanishing is the condition that the metric be Kählerian. Covariant derivatives are given by the usual formula

$$X_{\beta_1\cdots\beta_s|\gamma}^{\alpha_1\cdots\alpha_r} = \frac{\partial X_{\beta_1\cdots\beta_s}^{\alpha_1\cdots\alpha_r}}{\partial z^{\gamma}} + \sum_{t=1}^{r} X_{\beta_1\cdots\beta_s}^{\alpha_1\cdots\sigma\cdots\alpha_r} C_{\sigma\gamma}^{\alpha_t} - \sum_{t=1}^{s} X_{\beta_1\cdots\sigma\cdots\beta_s}^{\alpha_1\cdots\alpha_r} C_{\beta_t\gamma}^{\sigma}.$$

There are natural decompositions of a tensor into a sum of pure tensors of special types, those of a given type having components which vanish except for a particular pattern of Latin indices, for example, for all except the unbarred Latin indices. It is easy to see from the definition of covariant derivatives that if a tensor is pure and has only unbarred indices (example: $X_{j_1\cdots j_s}^{i_1\cdots i_s}$), then the components are analytic functions of the local coordinates in each coordinate system if and only if

Received October 7, 1957.

This research was supported by the United States Air Force through the AF Office of Scientific Research, Air Research and Development Command.