A GENERALIZATION OF BAGEMIHL'S THEOREM ON AMBIGUOUS POINTS

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Let f[z] be a function mapping the open unit disk $|z| < 1$ into the Riemann sphere. The point p on $|z| = 1$ is an *ambiguous point of* f (see [4]) if there exist two arcs A_1 and A_2, each with one end point at p, lying in the open disk except for p, and such that the limits of f at p along A_1 and along A_2 exist and are unequal. Bagemihl [1] proved a remarkable theorem: *even if* f *is not assumed to be continuous, it can have at most countably many points of ambiguity.* (Bagemihl's result was actually stronger; it is the case n = 2 of the Theorem below.) Bagemihl and Seidel [4] showed that every countable set on the unit circle is contained in the set of ambiguous points of some meromorphic function of bounded characteristic; and Lohwater and Piranian [7] have strengthened this by proving that every countable set on $|z| = 1$ is exactly the set of ambiguous points for some such function. It follows immediately from these results (or indeed from the existence of even one ambiguous point for a function in the unit disk) that if ambiguous points of a function in the (n - 1)-sphere of n-space are defined in terms of asymptotic behavior on arcs, in the obvious fashion, then there exist functions in the (n - 1)-sphere which have uncountably many ambiguous points.

But several other possible generalizations to higher dimensions suggest themselves. One might expect, for example, that the ambiguous points can not fill a cell on the (n - 1)-sphere. This possibility has been pretty thoroughly demolished by Bagemihl [2], [3], Piranian [9], and Church [5], who give examples of functions on the interior of the 2-sphere in E^3 for which the set of ambiguous points is a 2-cell; Church's example is a differentiable homeomorphism. In this note, I give a generalization in the spirit of Bagemihl's theorem.

I now define a "cell of disjoint cluster sets." Let D be a domain in Euclidean n-space E^n, and let $f: D \to S$ be a function from D into a topological space. A closed r-cell I in the boundary of D is an *r-cell of disjoint cluster sets for* f provided there exist two closed (r + 1)-cells J_1 and J_2, lying in D except for I (which is in the combinatorial boundary of each), such that the cluster set on I from J_1 of f does not meet the cluster set on I from J_2 of f; in other words, such that if $\{p_k\}$ is a sequence of points in J_1 - I, converging to a point p of I, and $\{q_k\}$ is a sequence of points in J_2 - I, converging to a point q of I, and lim $f(p_k)$, lim $f(q_k)$ exist in S, then they are not equal. (Clearly, this definition can be freed from the concept of sequences; however, I intend to apply it only where S is a compact metric space).

The examples mentioned in the second paragraph can easily be modified to show that there exist functions in the interior of the unit (n - 1)-sphere S^{n-1} in E^n such that there are uncountably many disjoint (n - 3)-cells of disjoint cluster sets that fill an (n - 1)-cell in S^{n-1} $(n > 2)$. The principal theorem of this paper is this, that there cannot be uncountably many disjoint (n - 2)-cells of disjoint cluster sets.

For completeness, we prove a lemma of a familiar type about the image space to be used.

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