A REPRESENTATION THEORY FOR MEASURES ON BOOLEAN ALGEBRAS

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1. INTRODUCTION

In an abstract (L)-space with weak unit (see [3], [6]), the characteristic elements constitute a Boolean σ-algebra. The countably additive measures on this algebra may be identified with the elements of the given (L)-space. These elements, as they reappear in the second adjoint space, are usually identified with certain Baire measures on the zero-dimensional, compact Hausdorff space associated with the first adjoint space of the given (L)-space.

The principal purpose of this paper is to represent measures on an arbitrary Boolean algebra as Baire measures on a zero-dimensional, compact Hausdorff space. Once this purpose is achieved, certain applications of the representation will be indicated. Of these, the most important is intended to be a conversion process, wherein considerations of measures and measurable functions with respect to a σ-field of sets are replaced by considerations of Baire measures and continuous functions with respect to a zero-dimensional, compact Hausdorff space.

A first key device to be used is a decomposition of a measure into a countably additive portion and a purely finitely additive portion. This device seems to be due to M. Woodbury [9]. It has been used by E. Hewitt and K. Yosida [10] and by H. Bauer [1]. A second important technique is the development of a Baire measure from a content. Familiarity with the explanation of this technique given in [4] will be assumed. As a third aid, free use will be made of the theory of (L)-spaces as developed in [3] and [6].

2. PRELIMINARY CONCEPTS

Let $\mathfrak{B}$ be an abstract Boolean algebra. Let o and e denote the null element and the unit element in $\mathfrak{B}$, while $\bar{a}$ indicates the complement of the element a with respect to these elements. Let $a \vee b$ and $a \wedge b$ denote the lattice operations as applied to a pair of elements in $\mathfrak{B}$. Frequent use will be made of the symbol $\Pi_{n=1}^{\infty} a_n$ as denoting the greatest lower bound in $\mathfrak{B}$ of a sequence $\{a_n\}$ of elements, when such a bound exists.

Let ϕ denote a real-valued function defined on $\mathfrak{B}$. Consider the three following properties, which might be postulated for ϕ.

(I) $\sup_{a \in \mathfrak{B}} |\phi(a)| < \infty$;

(II) $\phi(a \vee b) = \phi(a) + \phi(b)$ for all $a, b \in \mathfrak{B}$ with $a \wedge b = o$;

(III) $\lim_{n \to \infty} \phi(a_n) = 0$ for each nonincreasing sequence $\{a_n\}$ with $\Pi_{n=1}^{\infty} a_n = o$.

A real-valued function ϕ defined on $\mathfrak{B}$ and possessing properties (I) and (II) is said to be a *measure* on $\mathfrak{B}$. A measure with the additional property (III) is called a

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