EQUICONTINUITY AND COMPACTNESS IN LOCALLY CONVEX TOPOLOGICAL LINEAR SPACES

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1. INTRODUCTION

If E and F are locally convex spaces, and L(E, F) is the space of continuous linear mappings of E into F, the equicontinuous subsets of L(E, F) are of natural interest. Indeed, whether or not E is a t-space (*espace tonnelé* [3]) can be stated in terms of a property of such subsets. In this paper, the duality theory of linear spaces is applied systematically, by means of Lemma 2 below, to obtain characterizations of equicontinuity in L(E, F), in several cases in which E and F are given topologies different from the 'Mackey strong' topology τ. In particular, for the case of the topology k (Section 2), there is a natural connection between equicontinuity in L(E, F) and compactness in L(E, F) suitably topologized; the connection becomes especially simple if the spaces E and F satisfy certain restrictions in their τ-topologies. Sections 3, 4, and 5 all bear on the application of the theory in Section 6, where the compact subsets of the algebra of bounded operators on a Hilbert space are characterized in terms of equicontinuity, for several of the topologies studied by Dixmier [8]. Section 4 is devoted to a multiplicative property of equicontinuous subsets of L(E, E). The topological theorem of Section 5 is given more fully than its application to Section 6 requires, because of its possible intrinsic interest.

The symbol □ will denote the end of a proof or of some other expository unit, when paragraphing alone seems insufficient.

2. PRELIMINARIES

A pair of vector spaces E and E', over the same scalar (real or complex) field, are *in duality* if each is a separating set of linear functionals defined on the other. The value of a functional $x' \in E'$ at the point $x \in E$ will be denoted by (x', x). Everything to follow will be quite symmetric as between E and E'; it will therefore suffice to present all definitions and assertions in a one-sided way, the implication of a corresponding dual definition or assertion being understood.

Let θ denote the zero element of E. A topology on E will be named u if u is the set of all neighborhoods of θ, that is, the set of all sets having θ as an interior point. The topology u is *compatible* with the duality of E and E' if it is a locally convex topology on E for which E' is precisely the set of continuous linear functionals. E_u will then denote this locally convex space. If $A \subset E$, we say that $A^0 = \{x' \in E' \mid |(x', x)| \leq 1 \text{ for all } x \in A\}$ (see [3; 4; 5; 7] for such properties of this and other notions herein introduced and used without explitic reference). The weakest compatible topology on E, denoted by $\sigma(E, E')$ or simply by σ, has for a basis (at θ) the collection

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