ON SOLUTIONS OF THE EQUATION OF HEAT CONDUCTION

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1. INTRODUCTION

Suppose that $u = u(x, t)$ is defined over some domain D in the xt-plane. We say that u is parabolic in D if u is of class C^2 and $u_t = u_{xx}$ in D. Parabolic functions have many properties in common with harmonic functions. In this paper we use analogues of well-known theorems on harmonic and subharmonic functions to obtain a uniqueness theorem and some representation theorems for functions which are parabolic in the infinite strip $0 < t < c$.

We begin by introducing a class of subparabolic functions. For $x_1 < x_2$ and $t_1 < t_2$, let $R = R(x_1, x_2; t_1, t_2)$ denote the open rectangle bounded by the lines $x = x_1$, $x = x_2$, $t = t_1$, and $t = t_2$. Let S denote that part of the boundary for R which does not lie in the line $t = t_2$, and let $w = w(x, t)$ denote any function continuous in a domain D. When $R \cup S \subset D$, we define the function $M_R w = M_R w(x, t)$ as follows:

$$(1.1) \qquad \begin{cases} M_R w = w & \text{in } D - R, \\ M_R w & \text{is parabolic in } R, \\ M_R w & \text{is continuous at each point of } S. \end{cases}$$

Finally, we say that w is subparabolic in D if $w \leq M_R w$ for each rectangle R ($R \cup S \subset D$). (Compare [9].) When $P_0 = (x_0, t_0) \in R$, we have

$$(1.2) \qquad M_R w(P_0) = \int_S G(P_0, Q)\, w(Q)\, dS,$$

where the integration is taken over S, where

$$(1.3) \qquad \int_S G(P_0, Q)\, dS = 1,$$

and where

$$(1.4) \qquad G(P_0, Q) \geq 0$$

for $Q = (\xi, \tau) \in S$ [6]. Excepting the corners, we have strict inequality in (1.4) if and only if $t_1 \leq \tau \leq t_0$.

If w is subparabolic in D, then w is subparabolic in each subdomain of D. If u is parabolic, then $|u|^p$ is subparabolic for $p \geq 1$. A function u is parabolic if and only if the functions u and -u are subparabolic. Finally, both the sum and the upper envelope of a finite number of subparabolic functions are subparabolic.

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