DIFFERENTIATION ON MANIFOLDS WITHOUT A CONNECTION

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This note treats higher differentiations on a manifold in an invariant manner, without using any sort of connection. This is accomplished by considering a straightforward generalization of the ordinary Jacobian, which gives rise to new types of tensors. Some decomposition theorems are given, and speculations are made regarding other possible uses of the new tensors. For the sake of simplicity, only cases of lower order are treated.

Other work somewhat along these lines has been done by C. Ehresmann [1], [2] and A. Weil [3]; but their definitions and results are not required in what follows.

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The following conventions and notations will be observed. The manifolds considered allow at least three differentiations (real manifolds of class C^3). Sets of local coordinate variables are distinguished by different types of indices; for example, x^a, x^α, x^A stand for the coordinate functions of different coordinate systems. The symbols ∂_a, ∂_{ab}, ∂_α, $\partial_{\alpha\beta}$, $\cdots$ stand for the partial derivative operators $\partial/\partial x^a$, $\partial^2/\partial x^a \partial x^b$, $\partial/\partial x^\alpha$, $\partial^2/\partial x^\alpha \partial x^\beta$, $\cdots$. However, differentiation of the coordinate variables themselves will be denoted by capital D's; for example, $D^a_\alpha = \partial x^a/\partial x^\alpha$, $D^a_{\alpha\beta} = \partial^2 x^a/\partial x^\alpha \partial x^\beta$, $\cdots$. To avoid ambiguity, a symbol such as D^1_2 will never be used. Finally, repeated indices imply contraction, that is, summation over the repeated index: $D^a_\alpha D^\alpha_A = \Sigma_\alpha (\partial x^a/\partial x^\alpha)(\partial x^\alpha/\partial x^A) = \partial x^a/\partial x^A = D^a_A$.

Consider a generalized gradient $(\partial_a, \partial_{ab})$. This pair of differential operators has the transformation rule

$$(\partial_\alpha, \partial_{\alpha\beta}) = (\partial_a, \partial_{ab})\begin{pmatrix} D^a_\alpha & D^a_{\alpha\beta} \\ 0 & D^a_\alpha D^b_\beta \end{pmatrix} = (\partial_a D^a_\alpha, \partial_a D^a_{\alpha\beta} + \partial_{ab} D^a_\alpha D^b_\beta).$$

As the indices indicate, the "multiplication" of the entries of the two "matrices" means contraction. The transformation rule given by this generalized Jacobian operator $\begin{pmatrix} D^a_\alpha & D^a_{\alpha\beta} \\ 0 & D^a_\alpha D^b_\beta \end{pmatrix}$ is linear, homogeneous, and transitive; the last means that

$$\begin{pmatrix} D^a_\alpha & D^a_{\alpha\beta} \\ 0 & D^a_\alpha D^b_\beta \end{pmatrix}\begin{pmatrix} D^\alpha_A & D^\alpha_{AB} \\ 0 & D^\alpha_A D^\beta_B \end{pmatrix} = \begin{pmatrix} D^a_A & D^a_{AB} \\ 0 & D^a_A D^b_B \end{pmatrix},$$

which implies, in particular, that

$$\begin{pmatrix} D^a_\alpha & D^a_{\alpha\beta} \\ 0 & D^a_\alpha D^b_\beta \end{pmatrix}\begin{pmatrix} D^\alpha_c & D^\alpha_{cd} \\ 0 & D^\alpha_c D^\beta_d \end{pmatrix} = \begin{pmatrix} \delta^a_c & 0 \\ 0 & \delta^a_c \delta^b_d \end{pmatrix},$$

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