DENSE INVERSE LIMIT RINGS

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We are concerned here with certain ideals with finitely many generators, in linear algebras of a type studied in [1] and [8] and named $\mathcal{F}$-algebras by Michael [8]. Our main result is this: Let A be an $\mathcal{F}$-algebra with a unit element 1, and let J be a proper right ideal with finitely many generators, in A. Then there is a continuous homomorphism T of A into a Banach algebra B with unit such that T(J) lies in some proper right ideal of B. This is 4.2 below.

The assertion that B can be selected so as to be a Banach algebra (that is, a *complete* normed linear algebra), rather than merely a normed linear algebra, is important. In fact, the resulting weaker proposition is elementary, and we shall indicate briefly why it does not immediately imply our result 4.2. If T were a continuous homomorphism of A into a normed algebra N, and J_1 were the ideal generated by T(J), then J_1 would certainly be proper, but might be *everywhere-dense*. Thus if N were completed, J_1 might generate the improper ideal. On the other hand, in a Banach algebra with unit, each proper ideal is contained in a closed proper ideal.

In the special case in which A is commutative and J has a single generator (which is to say that J is the principal ideal generated by a single element a), we obtain a proposition which can be immediately deduced from the main theorems of the paper cited, namely [1, 7.1] and [8, 5.2]. The advance of the present work over these earlier papers comes from the technique presented below for controlling the variety of solutions $(x_1, \cdots, x_N)$ of the equation

1.1 $$a_1x_1 + \cdots + a_Nx_N = 1,$$

where $a_1, \cdots, a_N$ are given elements of some ring with unit. For $N > 1$, this variety exists even in the commutative case.

We call a system $\{a_1, \cdots, a_N\}$ *right regular* if 1.1 can be solved in the ring in question. If this ring is an algebra A with unit over the complex numbers, and $\lambda_1, \cdots, \lambda_N$ are complex numbers, while $a_1, \cdots, a_N$ belong to A, then $(\lambda_1, \cdots, \lambda_N)$ is said to belong to the *joint right spectrum* $\sigma(a_1, \cdots, a_n; A)$ of $(a_1, \cdots, a_N)$ if $\{a_1 - \lambda_1, \cdots, a_N - \lambda_N\}$ is *not* right regular. Our main result enables us to conclude that $\sigma(a_1, \cdots, a_N; A)$ is the union of the joint right spectra $\sigma(T(a_1), \cdots, T(a_N); B)$, where T is a continuous homomorphism of A (now assumed to be an $\mathcal{F}$-algebra with unit) into a Banach algebra B with unit, the union being over a countable family of such pairs (B, T). If, in addition, A is commutative, there is the following consequence. Let Δ be the class of all continuous homomorphism ζ of A on the complex numbers C (it is known that Δ has a natural one-to-one relation to the class of *closed* maximal ideals.) Then the joint spectrum $\sigma(a_1, \cdots, a_N; A)$ coincides with the image in complex N-space C^N of Δ obtained through the mapping

$$\zeta \to (\zeta(a_1), \cdots, \zeta(a_N)).$$

Michael has a discussion [8, 12.5] of the continuity of complex-valued homomorphisms of a commutative $\mathcal{F}$-algebra with unit; and in this discussion the

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