INDEPENDENT PERFECT SETS IN GROUPS

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INTRODUCTION. We shall consider locally compact abelian groups (written additively) in which every neighborhood of the identity 0 contains elements of infinite order; for brevity, we shall call such a group an I-group.

Hewitt has recently proved [1] that the convolution algebra of all regular, complex, bounded Borel measures on an I-group is not symmetric. This is an interesting extension of an earlier result of Šreĭder [3] concerning the measure algebra on the real line. The crux of Hewitt's extension is the construction, in every I-group, of a Cantor set (that is, a set homeomorphic to Cantor's ternary set) which is independent in the sense defined below. His construction depends on a fairly involved structure theorem and on the consideration of special cases (p-adic groups and complete direct sums of cyclic groups, in particular).

The present paper contains a much simpler construction of such sets. We use a modest amount of structure theory to reduce the problem to the case of a metric I-group, but in the metric case we simply imitate the usual construction of a Cantor set on the line as the intersection of a sequence of sets E_n which are unions of 2^n intervals.

DEFINITIONS. A subset E of an abelian group G is *independent* if the following is true: for every choice of distinct points $x_1, \cdots, x_j$ in E and of integers $n_1, \cdots, n_j$, not all 0, we have

$$n_1 x_1 + n_2 x_2 + \cdots + n_j x_j \neq 0.$$

By a *compact neighborhood* we shall mean the compact closure of a nonempty open set.

For $k = 1, 2, 3, \cdots$, G^k will denote the topological space which is the cartesian product of G with itself, taken k times; that is, $G^1 = G$, $G^k = G^{k-1} \times G$.

For any group-theoretic terms used, we refer to [2].

The main result of the paper is as follows:

THEOREM. *Every* I-*group contains an independent Cantor set.*

The proof will be in two steps:

STEP 1. *Every metric* I-*group contains an independent Cantor set.*

STEP 2. *Every* I-*group contains a closed subgroup which is a metric* I-*group.* (We use *metric* synonymously with *metrizable.*)

LEMMA 1. *Suppose* G *is an* I-*group*, $n_1, \cdots, n_k$ *are integers, not all equal to zero, and* E *is the set of all points* $(x_1, \cdots, x_k)$ *in* G^k *at which*

$$n_1 x_1 + n_2 x_2 + \cdots + n_k x_k \neq 0.$$

Then E *is a dense open subset of* G^k.

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