THE ASYMMETRY OF CERTAIN ALGEBRAS OF FOURIER-STIELTJES TRANSFORMS

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1. INTRODUCTION

Throughout the present paper, G will denote a locally compact Abelian group, and X its character group. We write the group operation as multiplication except in dealing with certain classical cases: no confusion should arise. (For all group-theoretic facts and terms not explained here, see [10].) The symbol R denotes the additive group of real numbers; T the multiplicative group of complex numbers of absolute value 1; N the additive group of all integers; Z(m) the additive group of integers modulo m (m = 2, 3, ⋯); and Δ_p the additive group of p-adic integers (p = 2, 3, 5, 7, 11, ⋯). For A and B in G, the symbol AB denotes the set $\{ab: a \in A,\ b \in B\}$.

Let $\mathfrak{B}$ (the Borel sets in G) be the smallest σ-algebra of subsets of G containing all compact sets. (For all set- and measure-theoretic terms and facts not explained here, see [3].) Let $\mathcal{M}(G)$ denote the set of all regular, countably additive, complex-valued, bounded Borel measures on G. For $\lambda \in \mathcal{M}(G)$, one can write

$$\lambda = \lambda_1 - \lambda_2 + i(\lambda_3 - \lambda_4)\,, \tag{1.1}$$

where λ_1, λ_2, λ_3, and λ_4 are nonnegative real measures in $\mathcal{M}(G)$, λ_1 is singular with respect to λ_2, and λ_3 is singular with respect to λ_4. Let $|\lambda| = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4$. We say that λ is concentrated on a set $E \in \mathfrak{B}$ if $|\lambda|(E') = 0$.

Let $\mathfrak{C}_\infty(G)$ denote the set of all continuous complex-valued functions on G each of which is arbitrarily small in absolute value outside of some compact set. It is well known that $\mathcal{M}(G)$ yields a concrete representation of the conjugate space of $\mathfrak{C}_\infty(G)$ (under the uniform norm in $\mathfrak{C}_\infty(G)$), the mapping

$$f \to \int_G f(x)\, d\lambda(x) \qquad (\lambda \in \mathcal{M}(G))$$

being the general bounded linear functional on $\mathfrak{C}_\infty(G)$. When each λ in $\mathcal{M}(G)$ is given its norm as a linear functional, $\mathcal{M}(G)$ becomes a complex Banach space.

It is also well known that $\mathcal{M}(G)$ is a Banach algebra under the operation of convolution:

$$\lambda * \mu(f) = \int_G \int_G f(xy)\, d\mu(y)\, d\lambda(x) \tag{1.2}$$

for $\lambda, \mu \in \mathcal{M}(G)$ and $f \in \mathfrak{C}_\infty(G)$ (see for example [8], 1.4.6). The value of the measure $\lambda * \mu$ for the Borel set E is

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