THE TOPOLOGIZATION OF A SEQUENCE SPACE BY TOEPLITZ MATRICES

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1. INTRODUCTION

It is our purpose to show how the convergence fields of certain regular Toeplitz matrices can serve as neighborhoods, in the topologization of a factor space of the space m of bounded sequences. We use two simple ideas: the principle of aping sequences, and the construction of Toeplitz matrices whose convergence fields are nontrivial but (in the sense in which this is possible) arbitrarily small.

The basic idea of the principle of aping sequences was introduced independently by Agnew [1, Theorem 4.1] and by Brundno [2]. But Agnew announced the principle only in a specialized form; and Brundno, who showed great skill in developing elaborate and highly fruitful applications, failed to make the principle explicit. Roughly, the idea can be stated as follows: *If a Toeplitz matrix* A *is regular* (or, more generally, if it has finite norm and if all its column limits exist), *then it transforms each pair of fairly similar bounded sequences into a pair of fairly similar bounded sequences.* Naturally, the expression "fairly similar" must in this context be defined in terms of the matrix A.

Terminology and Notation. By $A = (a_{nk})$ and $B = (b_{nk})$ we denote either Toeplitz matrices or the transformations which they represent. By a sequence

$$x = \{x_k\} = \{x(k)\}$$

we mean a sequence of complex constants, and by Ax the sequence $\{\Sigma_{k=0}^{\infty} a_{nk} x_k\}_{n=0}^{\infty}$, that is, the transform of x by A. Our matrices are regular, and the symbol Ax will never be used except where it is meaningful. If the sequence Ax converges to the number a, we write $Ax \to a$, and we say that A *evaluates* x *to* a.

The statement $x \sim y$ will mean that there exists a nonzero constant λ and a convergent sequence c such that $y_k = \lambda x_k + c_k$. Clearly, the relation $y \sim x$ is an equivalence relation; we denote by m/L the space of equivalence classes which it determines in m, and we use the letter X to represent the equivalence class to which x belongs. If $x \sim y$, then $Ax \sim Ay$, for every regular matrix A; we therefore use the symbol AX to denote the equivalence class to which Ax belongs. The set of bounded sequences evaluated by A is called the *convergence field of* A *in* m (or the *bounded convergence field of* A). By an immediate extension, we can speak of the convergence field of A in m/L.

If $\{k_r\}$ is an increasing sequence of positive integers, we say that $\{\xi_k\}$ *wanders slowly over* $\{k_r\}$ *provided*

$$\lim_{r\to\infty} \max_{k_r<k\leq k_{r+1}} |\xi(k) - \xi(k_r)| = 0.$$

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